Advances in Mechanics and Mathematics 27

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Evolution Inclusions and Variation Inequalities for Earth Data Processing III

Long-Time Behavior of Evolution Inclusions Solutions in Earth Data Analysis

Springer

Advances in Mechanics and Mathematics

Volume 27

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ISSN 1571-8689 ISSN 1876-9896 (electronic)
ISBN 978-3-642-28511-0 ISBN 978-3-642-28512-7 (eBook)
DOI 10.1007/978-3-642-28512-7
Springer Heidelberg New York Dordrecht London

Library of Congress Control Number: 2010936816

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

Lately, due to the intense investigation processes in economics, ecology, geophysics, synergy, and geoinformatics there is the necessity for more detailed study of nonlinear effects, new classes of mathematical models with nonlinear, nonsmooth, discontinuous, and multivalued dependence between determinative parameters of problems, control problems for nonlinear processes and fields, and long-range forecast problems for state functions of solutions for such problems. The Institute for Applied System Analysis (NAS) of Ukraine carries out fundamental research on such problems in the following areas:

1. Nonlinear analysis and control for classes of nonlinear geophysics processes and fields.
2. Properties of solutions of differential-operator inclusions and multivariation inequalities for Earth Data Analysis.
3. Theory of global and trajectory attractors for infinite-dimensional dynamical systems.

New qualitative and constructive results concerning properties of state functions for problems of analysis and control with nonsmooth interaction functions have been obtained.

Note that there exist classes of real problems with regular interaction functions that allow non-classical effects (for example equations of hydrodynamic type). We have also developed an abstract theory of geophysical problems. We propose the system mathematical instrument including the theory of differential-operator inclusions and multivariation inequalities for Earth Data Processes, theory of global and trajectory attractors for infinite-dimensional m-semiflows, methods for investigation of existence of solutions for control problems. Applying stated results to real problems, we provide long-range forecasts in cases where only the abstract theory for solutions is known (processes of piezoelectricity, viscoelasticity, thermodynamics, hydrodynamics etc.) (Fig. 1).

Moreover, the hypotheses of our theorems are close to the corresponding necessary conditions and the results of the theorems are close to the maximal ones. Let us consider the following example: when interaction function loses its smoothness it can be easily seen that finite fractal dimension decreases as well. This fact causes lack of stronger estimates (for example, exponential ones) in particular cases and lack of exponential attractors in the general case as well, as the loss of fractal dimension is the governing factor here.

These results again make the following fact very clear: it is impossible to adequately describe processes and fields with nonlinear and non-smooth interactions using smoothing techniques and the linearization method. Also, the results stimulate the development of adequate analogues of such theorems for non-smooth models.

The first two volumes concentrate mostly on development of constructive methods according to which solutions of general classes of such problems can be studied. Here we answer the questions concerning the long-time behavior of such problems. This fact poses new mathematical problems concerning adequate choice and analysis of a mathematical model (especially problems related to the smoothness of interaction functions). On the other hand, speaking about control and optimization problems, there arises a question concerning the choice of such admissible control sets that would allow more appropriate solution behavior on corresponding attractors. Therefore the new theory has arisen—the theory of extremal solutions to differential-operator problems

$$y' + A(y, y) \ni f. \tag{1}$$

Here, according to the method of artificial control, the new parameter is introduced, and the optimization problem

$$\begin{cases} y' + A(y, u) \ni f, \\ F(u, y) = \|y - u\|_W \to \inf, \ u \in U \subset W \end{cases} \tag{2}$$

takes its place instead of the initial evolutionary inclusion (1). Here W is a functional class of solutions possessing the necessary physical properties, $U \subset W$ is a space of artificial controls.

When problem (1) admits a classical solution in the given class, then it coincides with the solution of problem (2). Otherwise we have an accurate approximation of the initial solution, and, in particular, we can study the asymptotical behavior of all such solutions. Moreover, the results of our investigations are not conditional, unlike other known similar results (in particular, we apply our results to the 3-dimensional Navier–Stokes equation).

Our theory supplements well-known results for mathematical models of processes with smooth interaction functions. Rejecting smoothness in general cases, we guarantee general topological properties for attractors, which exist for smooth models. At the same time, we may lose properties related to estimates of the attractor dimension, though these properties are not natural ones for smooth models in general cases.

This book arose from seminars and lecture courses on multi-valued and nonlinear analysis and their geophysical application. These courses were delivered for rather different categories of learners in the National Technical University of

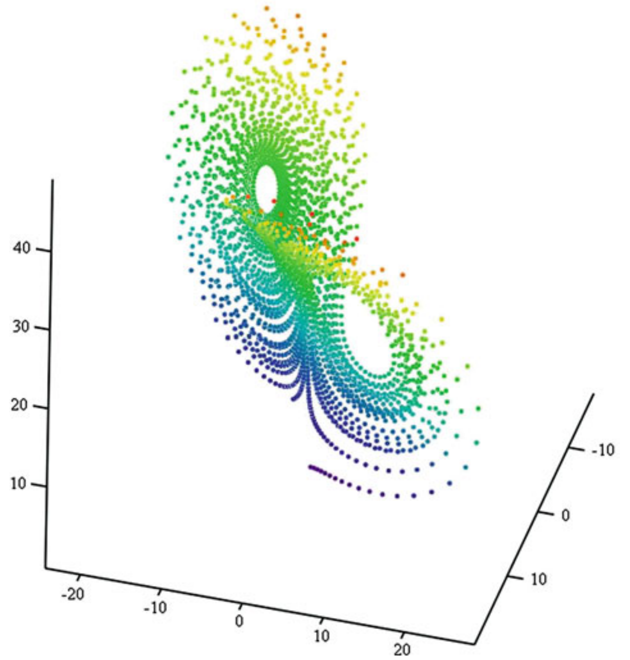

Fig. 1 Lorenz attractor (by M.Z. Zgurovsky, P.O. Kasyanov, O.V. Kapustyan, J. Valeró, N.V. Zadoianchuk, 2011)

Ukraine, "Kiev Polytechnic Institute", Taras Shevchenko National University of Kyiv, Universidad Miguel Hernández de Elche, University of Salerno etc. over 10 years. It is meant for a wide circle of mathematical and engineering students.

It is unnecessary to state that the pioneering work of such authors as J.M. Ball, V.S. Mel'nik, J.-L. Lions, V.V. Obukhovskii, N. Panagiotopoulos, N.S. Papageoriou, and I.V. Sergienko who created and developed the theory of mentioned problems, exerted a powerful influence on this book.

We are thankful to D. Gao, V.O. Gorban, I.M. Gorban, N.V. Gorban, V.I. Ivanenko, O.M. Novikov, V.V. Obuchovskii, M.O. Perestyuk, A.E. Shishkov for useful comments.

We are grateful to many students who attended our lectures while we were developing the notes for this volume.

We express our gratitude to O.P. Kogut and M.M. Perestyuk for their exceptional diligence in preparing the electronic version of the book and friendly help in linguistic issues.

We express our special gratitude to Kathleen Cass and Olena L. Poptsova for technical support in preparing the book.

Finally, we express our gratitude to the editors of the Springer Publishing House who worked with our books and everybody who took part in the preparation of the manuscript.

We apologize to people whose work has been missed inadvertently when making the references.

We will be grateful to readers for comments and corrections.

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March 2012

Contents

Introduction: Long-Time Behavior of Evolution Inclusion Solutions in Earth Data Analysis

Constructive and qualitative investigations of nonlinearized mathematical models have been activated lately. Such models of geophysical and socioeconomic processes and fields, under the proper interpretation of the derivative, can be reduced to differential-operator inclusions and multivariation inequalities in infinite-dimensional spaces. Among such models, we have to assign classes of unilateral problems, problems on a manifold with boundary and without boundary, problems with degenerates, objects of optimal control theory and filter theory, free-boundary problems, problems with delay, etc. Existing results concerned with the qualitative behavior of solutions of such problems, the theory of global and trajectory attractors for m-semiflows in infinite-dimensional spaces, chaos theory, and optimal control for distributed systems are based on nondegeneracy and closedness in corresponding topologies of the graph of the resolving operator for observable mathematical model. The proof of these results is the main problem when applying m-semiflows theory to real problems. Note that such properties of solutions for each equation are usually checked separately and this check, as a rule, is based on linearity or monotony of the main part of differential operator appeared in the problem. On the other hand, energetic extensions and Nemitsky operators for differential operators appeared in generalized settings of different problems of mathematical physics, stochastic partial differential equations, and problems with degenerates, as a rule, have (under corresponding choice of the phase space) common properties concerned with growth conditions (usually no more that polynomial), sign conditions, and pseudomonotony. In general case, under such restrictions for determinative parameters of initial problem, we succeed to prove only the existence of weak solutions of generated differential-operator inclusion or multivariation inequality, but not always this proof is constructive one. At the present moment, such objects have been strongly studied by many authors: Ball J.M., Zgurovsky M.Z., Mel'nik V.S., Kasyanov P.O., Kapustyan O.V., Valero J., Zadoianchuk N.V., Solonoukha O.M., Borisovich Yu. G., Gelman B.D., Kamenskii M.I., Mishkis A.D., Obukhovskii V.V., Kogut P.I., Kovalevsky O.A., Nicolosi Fr., Ansari Q.H., Khan Z., Yao J.-C., Aubin J.-P., Frankowska H., E.P. Avgerinos, N.S. Papageorgiou, Barbu V., Benchohra M., Ntouyas S.K., S. Carl, D. Motreanu, Hu S., Z. Liu, and others in

[1–12, 19–254], etc. At that, unnaturally strict conditions concerned with uniform coercivity, smoothness, linearity, etc., are considered for interaction functions of initial mathematical models. So, the adequate weakening of such technical conditions is undoubtedly an actual problem. It must be noted that when investigating important functional-topological properties of the resolving operator of differential-functional inclusions and multivariational inequalities, and validating of high-precision algorithm of search of approximate solutions, there appeared new problems concerned with the studying of new classes of energetic extensions and Nemitsky operators for differential operators of initial mathematical models, investigation of the structure of corresponding phase and extended phase spaces, proof of new theorems of embedding and approximation, basis theorem for such spaces, generalization of Fan Ky for multistrategies, and development of noncoercive theory for differential-operator inclusions with maps of pseudomonotone type. This book is a continuation of investigations in the sphere of nonlinear and multivalued analysis of distributed systems in infinite-dimensional spaces. So, the theme of the book concerned with creation of new theoretical instrument for qualitative and constructive investigation of a wide range of new, more precise, mathematical models of geophysical processes and fields with nonlinear, discontinuous, and multivalued interaction function, generalized solutions of which are solutions of differential-operator inclusions and evolution multivariation inequalities with noncoercive, in the classic sense, maps of pseudomonotone type, is an actual problem.

Multivalued dynamical systems have attracted the interest of many authors over the last years. These systems appear when the Cauchy problem of a differential equation does not posses the property of uniqueness. Hence, two or more solutions can exist corresponding to a given initial data. Due to this fact, a classical semigroup of operators cannot be defined, and other theory involving set-valued analysis is necessary. It should be noticed that in some cases, we know that we have really multivalued systems, but in others, we just do not know whether uniqueness holds or not. Hence, multivalued dynamical systems allow us not to stop when a proof of uniqueness fails for an equation, and the asymptotic behavior of solutions canbe studied no matter we have uniqueness or not.

Needless to say that this interest is mainly motivated by important applications in which this problem appears [1–12, 19–254]. Let us consider some motivational examples.

Example 1. We consider a linear viscoelastic body occupying the bounded domain Ω in $\mathbf{R}^N$ $(N = 2, 3)$ in a strainless state which is acted upon by volume forces and surface tractions and which may come in contact with a foundation on the part Γ_C of the boundary $\partial\Omega$. The boundary $\partial\Omega$ of the set Ω is supposed to be a regular one, and point data of $x \in \bar{\Omega}$ is considered in some fixed Cartesian system of coordinates. We assume that the body is endowed with short memory (cf. [74, p. 158]), that is, the state of the stress at the instant t depends only on the strain at the instant t and at the immediately preceding instants. In this case, the equation of state has the next form:

$$\sigma_{ij}(u) = b_{ijhk}\varepsilon_{kh}(u) + a_{ijhk}\frac{\partial}{\partial t}\varepsilon_{kh}(u), \quad i, j = 1, \ldots, N, \tag{3}$$

where $u : \Omega \times (0, +\infty) \to \mathbf{R}^N$ denotes the displacement field, $\sigma = \sigma(u)$ is the stress tensor, and $\varepsilon = \varepsilon(u)$ is the strain tensor, $\varepsilon_{hk}(u) = \frac{1}{2}(u_{k,h} + u_{h,k})$. The viscosity coefficients a_{ijhk} and the elasticity coefficients b_{ijhk} satisfy the well-known symmetry and ellipticity conditions. The dynamic behavior of the body is described by the equilibrium equation:

$$\sigma_{ij,j}(u) + f_i = \frac{\partial^2}{\partial t^2}u_i \quad \text{in} \quad \Omega \times (0, +\infty), \tag{4}$$

where $f = \{f_i\}_{i=1}^N \in L_2(\Omega; \mathbf{R}^N)$ denotes the density of body force. We suppose that the boundary $\partial\Omega$ is divided into three parts: Γ_D, Γ_N, and Γ_C. Exactly, let Γ_D, Γ_N, and Γ_C be disjoint sets and $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N} \cup \overline{\Gamma_C}$. We assume that $\Gamma_C \subset \partial\Omega$ is an open subset with positive surface measure (cf. [195, c. 196]). The displacements

$$u_i = 0 \text{ on } \Gamma_D \times (0, +\infty), \tag{5}$$

are prescribed on Γ_D, and surface tractions

$$S_i = \sigma_{ij}n_j = F_i \;\; (F_i = F_i(x)) \text{ on } \Gamma_N \times (0, +\infty). \tag{6}$$

are prescribed on Γ_N, where $F = \{F_i\}_{i=1}^N \in L_2(\Gamma_N; \mathbf{R}^N)$ denotes the vector of surface traction, $S = \{S_i\}_{i=1}^N$ is the stress vector on Γ_N, and $n = \{n_i\}_{i=1}^N$ is the outward unit normal to $\partial\Omega$.

On Γ_C, we specify nonmonotone multivalued boundary "reaction-velocity" conditions (cf. [87, 180, 188, 197] and references therein):

$$-S \in \partial j\left(x, \frac{\partial u}{\partial t}\right) \text{ on } \Gamma_C \times (0, +\infty), \tag{7}$$

where $j : \Gamma_C \times \mathbf{R}^N \to \mathbf{R}$ satisfies the next conditions:

1. $j(\cdot, \xi)$ is a measurable function for each $\xi \in \mathbf{R}^\mathbf{N}$ and $j(\cdot, 0) \in L_1(\Gamma_C)$.

2. $j(x, \cdot)$ is a locally Lipschitz function for each $x \in \Gamma_C$.

3. $\exists \bar{c} > 0$: $\|\eta\|_{\mathbf{R}^N} \le \bar{c}(1 + \|\xi\|_{\mathbf{R}^N})\ \forall x \in \Gamma_C, \forall \xi \in \mathbf{R}^N, \forall \eta \in \partial j(x, \xi)$,
   where for $x \in \Gamma_C$

$$\partial j(x, \xi) = \{\eta \in \mathbf{R}^N \mid (\eta, v)_{\mathbf{R}^N} \le j^0(x, \xi; v)\ \forall v \in \mathbf{R}^N\}$$

is the generalized gradient of the functional $j(x,\cdot)$ at point $\xi \in \mathbf{R}^N$,

$$j^0(x,\xi;v) = \varlimsup_{\zeta\to\xi,\, t\searrow 0} \frac{j(x,\zeta+tv)-j(x,\zeta)}{t}$$

is the generalized upper derivative of $j(x,\cdot)$ at point $\xi \in \mathbf{R}^N$ and the direction $v \in \mathbf{R}^N$.

Note that all nonconvex superpotential graphs from [188, Chap. 4.6], in particular, the functions j, defined as a minimum and as a maximum of quadratic convex functions, satisfy the upper considered conditions on Γ_C.

For the variational formulation of the problem (3)–(7), we set (cf. [180]): $H = L_2(\Omega;\mathbf{R}^N)$, $Z = H^\delta(\Omega;\mathbf{R}^N)$, $V = \{v \in H^1(\Omega;\mathbf{R}^N) : v_i = 0 \text{ on } \Gamma_D\}$, where $\delta \in \left(\frac{1}{2};1\right)$. Let $\forall u, v \in V$

$$\langle f_0, v\rangle_V = \int_\Omega f_i v_i dx + \int_{\Gamma_N} F_i v_i d\sigma(x),$$

$$a(u,v) = \int_\Omega a_{ijhk}\varepsilon_{ij}(u)\varepsilon_{ij}(v)dx,$$

$$b(u,v) = \int_\Omega b_{ijhk}\varepsilon_{ij}(u)\varepsilon_{ij}(v)dx,$$

$\bar\gamma : Z \to L_2(\partial\Omega;\mathbf{R}^N)$ be a trace operator, and $\bar\gamma^* : L_2(\partial\Omega;\mathbf{R}^N) \to Z^*$ be a conjugate operator,

$$\bar\gamma^* u(z) = \int_{\partial\Omega} u(x)\bar\gamma z(x)d\sigma(x), \quad z \in Z,\ u \in L_2(\partial\Omega;\mathbf{R}^N).$$

Let us consider a locally Lipschitz functional $J : L_2(\Gamma_C;\mathbf{R}^n) \to \mathbf{R}$,

$$J(z) = \int_{\Gamma_C} j(x,z(x))d\sigma(x), \quad z \in L_2(\Gamma_C;\mathbf{R}^n).$$

Then the interaction functions A_1, A_2, and B_0 can be defined by the next relations:

$$\forall z \in Z \quad A_2(z) = \bar\gamma^*\left(\partial J(\bar\gamma z)\right),$$

$$\forall u, v \in V \quad \langle A_1 u, v\rangle_V = a(u,v), \quad \langle B_0 u, v\rangle_V = b(u,v),\ A_0(u) = A_1 u + A_2(u).$$

If we supplementary have $\bar\alpha > \bar c\bar\beta^2\|\bar\gamma\|^2$, where $\bar\beta$ is the embedding constant of V into Z, $\bar\alpha$ is the constant from the ellipticity condition for a_{ijhk}, or

$$\forall x \in \Gamma_C,\ \forall \xi \in \mathbf{R}^N,\ \forall \eta \in \partial j(x,\xi) \quad (\eta,\xi)_{\mathbf{R}^N} \ge 0,$$

then from [180], it follows that the next condition hold, true:

(H_1) V, Z, H are Hilbert spaces; $H^* \equiv H$ and we have such chain of dense and compact embeddings:

$$V \subset Z \subset H \equiv H^* \subset Z^* \subset V^*.$$

(H_2) $f_0 \in V^*$.
(A_1) $\exists c > 0 : \forall u \in V, \forall d \in A_0(u)\ \|d\|_{V^*} \leq c(1 + \|u\|_V)$.
(A_2) $\exists \alpha, \beta > 0 : \forall u \in V, \forall d \in A_0(u)\ \langle d, u\rangle_V \geq \alpha\|u\|_V^2 - \beta$.
(A_3) $A_0 = A_1 + A_2$, where $A_1 : V \to V^*$ is linear, self-conjugated, positive operator and $A_2 : V \overset{\to}{\to} V^*$ satisfies such conditions:

a. There exists such Hilbert space Z, that the embedding $V \subset Z$ is dense and compact one and the embedding $Z \subset H$ is dense and continuous one.
b. For any $u \in Z$, the set $A_2(u)$ is nonempty, convex and weakly compact one in Z^*.
c. $A_2 : Z \overset{\to}{\to} Z^*$ is a bounded map, that is, A_2 converts bounded sets from Z into bounded sets in the space Z^*.
d. $A_2 : Z \overset{\to}{\to} Z^*$, is a demiclosed map, that is, if $u_n \to u$ in Z, $d_n \to d$ weakly in Z^*, $n \to +\infty$, and $d_n \in A_2(u_n)\ \forall n \geq 1$ then $d \in A_2(u)$.

(B_1) $B_0 : V \to V^*$ is a linear self-conjugated operator.
(B_2) $\exists \gamma > 0 : \langle B_0 u, u\rangle_V \geq \gamma\|u\|_V^2$.

Here, $\langle \cdot, \cdot\rangle_V : V^* \times V \to \mathbf{R}$ is the duality in $V^* \times V$, coinciding on $H \times V$ with the inner product $(\cdot, \cdot)$ in Hilbert space H.

Note that from (A_1)–(A_3) [180, 253], it follows that the map A_0 satisfies such condition:

$(A_3)'$ $A_0 : V \overset{\to}{\to} V^*$ is (generalized) λ_0-pseudomonotone, i.e.:

a. For any $u \in V$ the set $A_0(u)$ is nonempty, convex, and weakly compact one in V^*.
b. If $u_n \to u$ weakly in V, $n \to +\infty$, $d_n \in A_0(u_n)\ \forall n \geq 1$, and $\overline{\lim\limits_{n\to\infty}} \langle d_n, u_n - u\rangle_V \leq 0$, then $\forall \omega \in V\ \exists d(\omega) \in A_0(u)$:

$$\underline{\lim}_{n\to+\infty} \langle d_n, u_n - \omega\rangle_V \geq \langle d(\omega), u - \omega\rangle_V.$$

c. The map A_0 is upper semicontinuous one that acts from an arbitrary finite-dimensional subspace of V into V^*, endowed with weak topology.

Thus, we investigate the dynamic of all weak solutions of the second-order nonlinear autonomous differential-operator inclusion

$$y''(t) + A_0(y'(t)) + B_0(y(t)) \ni f_0, \tag{8}$$

as $t \to +\infty$, which are defined as $t \geq 0$, where parameters of the problem satisfy conditions (H_1), (H_2), (A_1)–(A_3), (B_1)–(B_2).

As a *weak solution* of the evolution inclusion (8) on the interval $[\tau, T]$, we consider such pair of elements $(u(\cdot), u'(\cdot))^T \in L_2(\tau, T; V \times V)$, that for some $d(\cdot) \in L_2(\tau, T; V^*)$

$$\begin{aligned} d(t) \in A_0(u'(t)) \quad &\text{for almost every (a.e.)} \quad t \in (\tau, T),\\ &- \int_\tau^T (\zeta'(t), u'(t))dt + \int_\tau^T \langle d(t), \zeta(t)\rangle_V dt \\ &+ \int_\tau^T \langle B_0 u(t), \zeta(t)\rangle_V dt = \int_\tau^T \langle f_0, \zeta(t)\rangle_V \ \ \forall \zeta \in C_0^\infty([\tau, T]; V), \end{aligned} \tag{9}$$

where u' is the derivative of the element $u(\cdot)$ in the sense of the space of distributions $\mathscr{D}^*([\tau, T]; V^*)$.

As a *generalized solution* of the problem (3)–(7) we consider the weak solution of the corresponding problem (8). This definition is coordinated with Definition 3 from [180].

We have to note that abstract theorems on existence of solutions for such problems as the problem (8) and the optimal control problems for weaker conditions for parameters of problems are considered in works [180, 241, 242, 252, 253]. Here, we consider Problem 2 from [180], for which we can (as follows from results of the given paper) have not only the abstract result on existence of weak solution, but we can investigate the behavior of all weak solutions as $t \to +\infty$ in the phase space $V \times H$ and study the structure of the global and trajectory attractors.

Example 2. We consider a mathematical model which describes the contact between a piezoelectric body and a foundation [161]. The physical setting is formulated as in [161]. We consider a plane electro-elastic material which in its undeformed state occupies an open bounded subset Ω of $\mathbf{R}^d$, $d = 2$. We agree to keep this notation since the mathematical results hold for $d \geq 2$. The boundary Γ of the piezoelectric body Ω is assumed to be Lipschitz continuous. We consider two partitions of Γ. A first partition is given by two disjoint measurable parts Γ_D and Γ_N such that $m(\Gamma_D) > 0$, and a second one consists of two disjoint measurable parts Γ_a and Γ_b such that $m(\Gamma_a) > 0$. We suppose that the body is clamped on Γ_D, so the displacement field vanishes there. Moreover, a surface tractions of density g act on Γ_N, and the electric potential vanishes on Γ_a.

The body Ω is lying on another medium (the so-called support) which introduce frictional effects. The interaction between the body and the support is described, due to the adhesion ar skin friction, by a nonmonotone possibly multivalued law between the bonding forces and the corresponding displacements. In order to formulate the skin effects, we suppose (following [188, 198]) that the body forces of density f consist of two parts: f_1 which is prescribed external loading and f_2 which is the reaction of constrains introducing the skin effects, that is, $f = f_1 + f_2$. Here, f_2 is a possibly multivalued function of the displacement u. We consider the reaction-displacement law of the form

$$-f_2(x,t) \in \partial j(x,t,u(x,t)) \text{ in } Q,$$

where $Q = \Omega \times (0,T)$, $(0,T)$ denotes a finite time interval, $j : Q \times \mathbf{R}^2 \to \mathbf{R}$ is locally Lipschitz function in its last variable and ∂j represents the Clarcke subdifferential.

The governing equations of piezoelectricity consist (see, e.g. [60, 99, 100, 163]) of the equation of motion, equilibrium equation, constitutive relations, strain-displacement, and electric field-potential relations. We suppose that the accelerations in the system are not negligible and therefore the process is dynamic.

The equation of motion for the stress field and *the equilibrium equation* for the electric displacement field are, respectively, given by

$$\rho u'' - \mathrm{Div}\sigma = \rho f - \gamma u' \text{ in } Q,$$

$$\mathrm{div} D = 0 \text{ in } Q,$$

where ρ is the constant mass density (normalized as $\rho = 1$), $\gamma \in L_\infty(\Omega)$ is a nonnegative function characterizing the viscosity (damping) of the medium, $\sigma : Q \to S_d$, $\sigma = (\sigma_{ij})$, and $D : \Omega \to \mathrm{R}^d$, $D = (D_i)$, $i, j = 1, \ldots, d$ represent the stress tensor and the electric displacement field, respectively. Here, S_d is the linear space of second-order symmetric tensors on R^d with the inner product and the corresponding norm $\sigma : \tau = \sum_{ij} \sigma_{ij}\tau)ij$, $\|\tau\|^2_{S_d} = \tau : \tau$, respectively. Recall also that Div is the divergence operator for tensor valued functions given by $\mathrm{Div}\sigma = (\sigma_{ij,j})$ and div stands for the divergence operator for vector-valued functions, that is, $\mathrm{div} D = (D_{i,i})$.

The stress-charge form of piezoelectric constitutive relations describes the behavior of the material and are following

$$\sigma = \mathscr{A}\varepsilon(u) - \mathscr{P}E(\varphi) \text{ in } Q \text{ (converse effect)},$$

$$D = \mathscr{P}\varepsilon(u) + \mathscr{B}E(\varphi) \text{ in } Q \text{ (direct effect)},$$

where $\mathscr{A} : \Omega \times S_d \to S_d$ is a linear elasticity operator with the elasticity tensor $a = (a_{ijkl})$, $\mathscr{P} : \Omega \times S_d \to \mathbf{R}^d$ is a linear piezoelectric operator represented by the piezoelectric coefficients $p = (p_{ijk})$, $i, j, k = 1, \ldots d$ (third-order tensor field), $\mathscr{P}^T : \Omega \times \mathbf{R}_d \to S_d$ is its transpose represented by $p^T = (p^T_{ijk}) = (p_{kij})$, and $\mathscr{B} : \Omega \times \mathbf{R}^d \to \mathbf{R}^d$ is a linear electric permittivity operator with the dielectric constants $\beta = (\beta_{ij})$ (second-order tensor field). The decoupled state (purely elastic and purely electric deformations) can be obtained by setting the piezoelectric coefficients $p_{ijk} = 0$. The elasticity coefficients $a(x) = (a_{ijkl}(x))$, $i, j, k, l = 1, \ldots d$ (fourth-order tensor field) are functions of position in a nonhomogeneous material. We use here notation p^T to denote the transpose of the tensor p given $p\tau \cdot v = \tau : p^T v$ for $\tau \in S_d$ and $v \in \mathbf{R}^d$.

The elastic strain-displacement and electric field-potential relations are given by

$$\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T) \text{ in } Q,$$
$$E(u) = -\nabla\varphi \text{ in } \Omega,$$

where $\varepsilon(u) = (\varepsilon_{ij}(u))$ and $E(\varphi) = (E_i(\varphi))$ denote the linear strain tensor and the electric vector field, respectively. Here, $u : Q \to \mathbf{R}^d$, $u = (u_i)$, $i = 1, \dots, d$ and $\varphi : \Omega \to \mathbf{R}$ are the displacement vector field and the electric potential (scalar field), respectively.

Denoting by u_0 and u_1, the initial displacement and initial velocity, respectively, the classical formulation of the mechanical model can be stated as follows: find a displacement field $u : Q \to \mathbf{R}^d$ and an electric potential $\varphi : \Omega \to \mathbf{R}^d$ such that

$$u'' - \mathrm{Div}\sigma = f_1 + f_2 - \gamma u' \text{ in } Q \tag{10}$$

$$\mathrm{div} D = 0 \text{ in } Q \tag{11}$$

$$\sigma = \mathscr{A}\varepsilon(u) + \mathscr{P}^T\nabla\varphi \text{ in } Q, \tag{12}$$

$$D = \mathscr{P}\varepsilon(u) - \mathscr{B}\nabla\varphi \text{ in } Q, \tag{13}$$

$$u = 0 \text{ on } \Gamma_D \times (0, T) \tag{14}$$

$$\sigma n = g \text{ on } \Gamma_D \times (0, T) \tag{15}$$

$$\varphi = 0 \text{ on } \Gamma_a \times (0, T) \tag{16}$$

$$D \cdot n = 0 \text{ on } \Gamma_b \times (0, T) \tag{17}$$

$$-f_2(x, t) \in \partial j(x, t, u(x, t)) \text{ in } Q \tag{18}$$

$$u(0) = u_0,\ u'(0) = u_1 \text{ in } \Omega, \tag{19}$$

where n denotes the outward unit normal to Γ. The above problem with no skin effects (i.e., with $j = 0$) and with $\Gamma_N = \emptyset$, $\gamma(x) = \gamma$, γ being a nonnegative constant, has been studied by Cimatti in [60] who obtained existence and uniqueness of weak solutions by using the Galerkin method and the theory of semigroups. In [161] it is investigated the dynamic problem (10)–(19) which includes nonmonotone skin effects. Because of the Clarke subdifferential in (18), the problem will be formulated as a hemivariational inequality, and then, it will be embedded into a more general class of second-order evolution inclusions. Due to the multivalued term in the problem, the uniqueness of weak solutions is not expected.

In [161], it is given a simple one-dimensional example of the multivalued condition (18) for which superpotential satisfies the hypothesis $H(j)$ of Sect. 4 of [161]. Let $j : Q \times \mathbf{R} \to \mathbf{R}$ be defined as a minimum of two convex functions, that is, $j(x, t, s) = h(x, t) \min\{j_1(s), j_2(s)\}$ for $(x, t) \in Q$ and $s \in \mathbf{R}$, where $h \in L_\infty(Q)$, $j_1(s) = as^2$ and $j_2(s) = \frac{a}{2}(s^2 + 1)$ with $a > 0$. Then

$$\partial j(x,t,s) = h(x,t) \times \begin{cases} as & \text{if } s \in (-\infty,-1) \cup (1,+\infty) \\ 2as & \text{if } s \in (-1,1) \\ [a,2a] & \text{if } s = 1 \\ [-2a,-a] & \text{if } s = -1. \end{cases}$$

The model example can be modified to obtain nonmonotone zig–zag relations which describe the adhesive contact laws for a granular material and a reinforced concrete, for example, the stick-slip law and the fiber bundle model law (see [198], Sects. 2.4 and 7.2 of [196] and Sect. 4.6 of [188]).

Another example which satisfies $H(j)$ is a superpotential of d.c. (difference of convex functions) type, that is, $j(s) = j_1(s) - j_2(s)$, where $j_1, j_2 : \mathbf{R} \to \mathbf{R}$ are convex functions. We refer to Example 13 of [183] for more details.

We now turn to the variational formulation of the problem (10)–(19). We introduce the spaces for the displacement and electric potential:

$$V = \{v \in H^1(\Omega;\mathbf{R}^d) : v = 0 \text{ on } \Gamma_D\}, \tag{20}$$
$$\Phi = \{\varphi \in H^1(\Omega) : \varphi = 0 \text{ on } \Gamma_a\}$$

which are closed subspaces of $H^1(\Omega;\mathbf{R}^d)$ and $H^1(\Omega)$, respectively. Let $H = L_2(\Omega;\mathbf{R}^d)$ and $\mathcal{H} = L_2(\Omega;S_d)$ be Hilbert spaces equipped with the inner products $\langle u,v\rangle_H = \int_\Omega u \cdot v dx$, $\langle \sigma,\tau\rangle_{\mathcal{H}} = \int_\Omega \sigma : \tau dx$. Then the spaces (V,H,V^*) form an evolution triple of spaces. On V, we consider the inner product and the corresponding norm given by $\langle u,v\rangle_V = \langle \varepsilon(u),\varepsilon(v)\rangle_{\mathcal{H}}$, $\|v\|_V = \|\varepsilon(v)\|_{\mathcal{H}}$ for $u,v \in V$. From the Korn inequality $\|v\|_{H^1(\Omega;\mathbf{R}^d)} \le C\|\varepsilon(v)\|_{\mathcal{H}}$ for $v \in V$ with $C>0$, it follows that $\|\cdot\|_{H^1(\Omega;\mathbf{R}^d)}$ and $\|\cdot\|_V$ are equivalent norms on V. Thus, $(V,\|\cdot\|_V)$ is a Hilbert space. On Φ, we consider the inner product $(\varphi,\psi)_\Phi = (\varphi,\psi)_{H^1(\Omega)}$ for $\varphi,\psi \in \Phi$. Then $(\Phi,\|\cdot\|_\Phi)$ is also a Hilbert space.

Assuming sufficient regularity of the functions involved in the problem (10)–(19), multiplying (10) by $v \in V$ and using integration by parts, we have

$$\langle u''(t),v\rangle + \langle \sigma(u),\varepsilon(v)\rangle_{\mathcal{H}} - \int_\Gamma \sigma n \cdot v d\Gamma(x) = \langle f_1(t) + f_2(t), v\rangle - \langle \gamma u'(t), v\rangle$$

for a.e. $t \in (0,T)$. Since, by (15), we have $\int_\Gamma \sigma n \cdot v d\Gamma = \int_{\Gamma_N} g(t) \cdot v d\Gamma$ and (18) implies

$$-\int_\Omega f_2(x,t) \cdot v(x) dx \le \int_\Omega j^0(x,t,u(x,t);v(x)) dx \text{ for a.e. } t \in (0,T),$$

we obtain

$$\langle u''(t),v\rangle + \langle \gamma u'(t),v\rangle + \langle \sigma(u),\varepsilon(v)\rangle_{\mathcal{H}} + \int_\Omega j^0(x,t,u(x,t);v(x))dx \ge \langle F(t),v\rangle \tag{21}$$

where

$$\langle F(t), v\rangle = \langle f_1(t), v\rangle + \int\limits_{\Gamma_N} g(t)\cdot v d\Gamma \ \text{ for } \ v \in V.$$

Let $\psi \in \Phi$. From (11), again by using integration by parts and the condition (17), we have

$$-\langle D, \nabla\psi\rangle_H = 0. \tag{22}$$

Now inserting (12) into (21) and (13) into (22), we obtain the following variational formulation: find $u \in C(0,T;V) \cap C^1(0,T;V)$ and $\varphi \in L_2(0,T;H)$ such that $u'' \in \mathscr{V}$, where $\mathscr{V} = L_2(0,T;V^*)$ and

$$\begin{cases} \langle u''(t), v\rangle + \langle \gamma u'(t), v\rangle + \langle \mathscr{A}\varepsilon(u), \varepsilon(v)\rangle_{\mathscr{H}} + \langle \mathscr{P}^T\nabla\varphi, \varepsilon(v)\rangle_{\mathscr{H}} + \\ + \int\limits_{\Omega} j^0(x,t,u;v)dx \geq \langle F(t), v\rangle \quad \text{a.e. } t, \ \text{for all } v \in V \\ \langle \mathscr{B}\nabla\varphi, \nabla\psi\rangle_H = \langle \mathscr{P}\varepsilon(u), \nabla\psi_H \quad \text{a.e. } t, \ \text{for all } \ \psi \in \Phi \\ u(0) = u_0, \ u'(0) = u_1. \end{cases} \tag{23}$$

We need the following hypotheses for the constitutive tensors:

$\underline{H(a)}$: The elasticity tensor field $a = (a_{ijkl})$ satisfies $a_{ijkl} \in L_\infty(\Omega)$, $a_{ijkl} = a_{klij}$, $a_{ijkl} = a_{jikl}$, $a_{ijkl} = a_{ijlk}$ and $a_{ijkl}(x)\tau_{ij}\tau_{kl} \geq \alpha\tau_{ij}\tau_{ij}$ for a.e $x \in \Omega$ and all $\tau = (\tau_{ij}) \in S_d$ with $\alpha > 0$.
$\underline{H(p)}$: The piezoelectric tensor field $p = (p_{ijk})$ satisfies $p_{ijk} = p_{ikj} \in L_\infty(\Omega)$.
$\underline{H(\beta)}$: The dielectric tensor field $\beta = (\beta_{ij})$ satisfies $\beta_{ij} = \beta_{ji} \in L_\infty(\Omega)$ and $\beta_{ij}(x)\xi_i\xi_j \geq m_\beta|\xi|^2_{\mathbf{R}^d}$ for a.e. $x \in \Omega$ and all $\xi = (\xi_i) \in \mathbf{R}^d$ with $m_\beta > 0$.

We define the following bilinear forms $a : V \times V \to \mathbf{R}$, $b : V \times \Phi \to \mathbf{R}$, $b^T : \Phi \times V \to \mathbf{R}$, and $c : \Phi \times \Phi \to \mathbf{R}$ by

$$a(u,v) = \int\limits_{\Omega} a_{ijkl}(x)\frac{\partial u_k}{\partial x_l}\frac{\partial v_i}{\partial x_j}dx \quad \text{for} \quad u, v \in V,$$

$$b(u,\varphi) = \int\limits_{\Omega} p_{ijk}(x)\frac{\partial u_i}{\partial x_j}\frac{\partial \varphi}{\partial x_k}dx \quad \text{for} \quad u \in V, \quad \varphi \in \Phi,$$

$$b^T(\varphi,u) = \int\limits_{\Omega} p_{kij}(x)\frac{\partial \varphi}{\partial x_k}\frac{\partial u_i}{\partial x_j}dx \quad \text{for} \quad \varphi \in \Phi, \quad u \in V,$$

$$c(\varphi,\psi) = \int\limits_{\Omega} \beta_{ij}(x)\frac{\partial \varphi}{\partial x_i}\frac{\partial \psi}{\partial x_j}dx \quad \text{for} \quad \varphi, \psi \in \Phi.$$

Then we have

$$a(u, v) = \langle \mathscr{A}\varepsilon(u), \varepsilon(v)\rangle_{\mathscr{H}} \quad \text{for } u, v \in V,$$

$$b(u, \varphi) = \langle \mathscr{P}\varepsilon(u), \nabla\varphi\rangle_H \quad \text{for } u \in V,\ \varphi \in \Phi,$$

$$b^T(\varphi, u) = \langle \mathscr{P}\nabla\varphi, \varepsilon(u)\rangle_{\mathscr{H}} \quad \text{for } \varphi \in \Phi,\ u \in V,$$

$$c(\varphi, \psi) = \langle \mathscr{B}\nabla\varphi, \nabla\psi\rangle_H \quad \text{for } \varphi, \psi \in \Phi,$$

where the elasticity operator $\mathscr{A} : \Omega \times S_d \to S_d$ is given by $\mathscr{A}(x, \varepsilon) = a(x)\varepsilon$, $a(x) = (a_{ijkl}(x))$, the piezoelectric operator $\mathscr{P} : \Omega \times S_d \to \mathbf{R}^d$ is given by $\mathscr{P}(x, \varepsilon) = p(x)\varepsilon$, $p(x) = (p_{ijk}(x))$, the transpose to $\mathscr{P}$, $\mathscr{P}^T : \Omega \times \mathbf{R}^D \to S_d$ is given by $\mathscr{P}^T(x, \xi) = p^T(x)\xi$, $p^T(x) = (p^T_{ijk}(x)) = (p_{kij})$, and the electric permittivity operator $\mathscr{B} : \Omega \times \mathbf{R}^d \to \mathbf{R}^d$ is defined by $\mathscr{B}(x, \xi) = \beta(x)\xi$, $\beta(x) = (\beta_{ij(x)})$.

Using the above notation, the problem (23) is formulated as follows: find $u \in C(0, T; V) \cap C^1(0, T; H)$ and $\varphi \in L_2(0, T; H)$ such that $u'' \in \mathscr{V}^*$ and

$$\begin{cases} \langle u''(t), v\rangle + \langle \gamma u'(t), v\rangle + a(u(t), v) + b^T(\varphi(t), v) + \\ + \int\limits_\Omega j^0(x, t, u; v)dx \geq \langle F(t), v\rangle \quad \text{a.e. } t, \quad \text{for all } v \in V \\ c(\varphi(t), \psi) = b(u(t), \psi) \quad \text{a.e. } t, \quad \text{for all } \psi \in \Phi \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{24}$$

The problem (24) is a system coupled with the hemivariational inequality for the displacement and a time-dependent stationary equation for the electric potential. We need now some auxiliary results and notation.

We remark that under hypotheses $H(p)$ and $H(\beta)$, for any $z \in V$, there exists a unique element $\varphi_z \in \Phi$ such that

$$c(\varphi_z, \psi) = b(z, \psi) \quad \text{for all} \quad \psi \in \Phi$$

and the map $C : V \to \Phi$ given by $Cz = \varphi_z$ is linear and continuous.

As a corollary, we obtain the following result: if $H(p)$, $H(\beta)$ hold and $u \in \mathscr{V}$, where $\mathscr{V} = L_2(0, T; V)$, then the problem

$$\begin{cases} \text{find } \varphi \in L_2(0, T; \Phi) \quad \text{such that} \\ c(\varphi(t), \psi) = b(u(t), \psi) \quad \text{for a.e. } t \in (0, T), \quad \text{all } \psi \in \Phi \end{cases}$$

admits a unique solution $\varphi \in L_2(0, T; \Phi)$ and $\|\varphi\|_{L_2(0,T;\Phi)} \leq c\|u\|_{\mathscr{V}}$ with $c > 0$. For a.e. $t \in (0, T)$, we have $\varphi(t) = Cu(t)$, where the operator C is defined in Lemma 3.1 of [161].

Next, since for every $\varphi \in \Phi$, the linear form $v \to b^T(\varphi, v)$ is continuous on V, so there exists a unique element $B\varphi \in V^*$ such that $b^T(\varphi, v) = \langle B\varphi, v\rangle_{V^* \times V}$ for all $v \in V$ and $B \in \mathscr{L}(\Phi, V^*)$. We observe that

$$b^T(\varphi, v) = \langle \mathscr{P}\nabla\varphi, \varepsilon(v)\rangle_{\mathscr{H}} = \int_\Omega \mathscr{P}^T\nabla\varphi : \varepsilon(v)dx \tag{25}$$
$$= \int_\Omega \mathscr{P}\varepsilon(v)\cdot\nabla\varphi dx = \langle \mathscr{P}\varepsilon(v), \nabla\varphi\rangle_H = b(v, \varphi) \text{ for all } v \in V, \text{ and } \varphi \in \Phi.$$

Similarly, we introduce the operator $A \in \mathscr{L}(V, V^*)$ such that $a(u, v) = \langle Au, v\rangle$ for all $u, v \in V$.

We are now in a position to reformulate the system (24). Since for a fixed $u \in \mathscr{V}$, the second equation in (24) is uniquely solvable (cf. Corollary 1 in [161]), we have

$$b^T(\varphi(t), v) = \langle B\varphi(t), v\rangle = \langle BCu(t), v\rangle \quad \text{for all } v \in V \quad \text{and a.e. } t \in (0, T).$$

Thus, the problem (24) takes the form: find $u \in C(0, T; V) \cap C^1(0, T; H)$ such that $u'' \in \mathscr{V}$ and

$$\begin{cases} \langle u''(t) + Ru'(t) + Gu(t), v\rangle + \int_\Omega j^0(x, t, u; v)dx \geq \langle F(t), v\rangle \\ \text{a.e. } t, \quad \text{for all } v \in V \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases} \tag{26}$$

where $R : H \to V^*$ and $G : V \to V^*$ are given by $Rv = \gamma v$ for $v \in H$ and $Gv = Av + BCv$ for $v \in V$, respectively.

The existence of solutions to the hemivariational inequality (26) will be a consequence of a more general result provided in [161]. We remark that operators R and G satisfy such properties: if $\gamma \in L_\infty(\Omega)$, $\gamma \geq 0$, then the operator $R : H \to V^*$ defined by $Rv = \gamma v$ is linear continuous and $\langle Rv, v\rangle_{V^*\times V} \geq 0$ for all $v \in V$. Under the hypotheses $H(a)$, $H(p)$ and $H(\beta)$, the operator $G : V \to V^*$ defined by $G = A + BC$ is linear, bounded, symmetric, and coercive.

Finally, we obtain the following second-order evolution inclusion: find $u \in C(0, T; V) \cap C^1(0, T; H)$ such that $u'' \in \mathscr{V}^*$ and

$$\begin{cases} u''(t) + Ru'(t) + Gu(t) + \partial J(t, u(t)) \ni F(t) \quad \text{a.e.} \quad t \in (0, T) \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{27}$$

We need the following hypotheses:

$\underline{H(R)}$: $R : H \to V^*$ is a linear bounded operator such that $R|_v$ is nonnegative (i.e., $\langle Rv, v\rangle_{V^*\times V} \geq 0$ for all $v \in V$).
$\underline{H(G)}$: $G \in \mathscr{L}(V, V^*)$ is symmetric and coercive operator.
$\underline{H(J)}$: $J : (0, T) \times H \to \mathbf{R}$ is a function such that
(i) $J(\cdot, v)$ is measurable for all $v \in H$.
(ii) $J(t, \cdot)$ is locally Lipschitz for a.e. $t \in (0, T)$.
(iii) $|\partial J(t, v)| \leq \tilde{c}(1 + |v|)$ for a.e. $t \in (0, T)$ and $v \in H$ with $\tilde{c} > 0$; where $\partial J(t, v)$ denotes the Clarke subdifferential of $J(t, \cdot)$ at a point $v \in H$.
$\underline{(H_0)}$: $F \in \mathscr{H}$, $u_0 \in V$, $u_1 \in H$.

In [161], it is proved that if hypotheses $H(R)$, $H(G)$, $H(J)$, and (H_0) hold, then the problem (27) has a solution.

We investigate a long-time behavior of all weak solutions of the problem (27) under similar, but some stronger (providing a dissipation) conditions. In particular, we study the structure of the global and trajectory attractors.

Example 3. Let $\Omega \subset \mathrm{R}^3$ be a bounded domain with a smooth boundary, $\nu > 0$. For $y : (0,T)\times\Omega \to \mathrm{R}^n$, let us consider the three-dimensional Navier–Stokes problem

$$\frac{\partial y}{\partial t} - \nu\Delta y + \sum_{i=1}^{3} y_i \frac{\partial y}{\partial x_i} = f(t,x) - \nabla p, \tag{28}$$

$$\operatorname{div} y = 0, \tag{29}$$

$$y\,|_{\Sigma} = 0, \quad \sum = (0,T)\times\partial\Omega, \tag{30}$$

$$y|_{t=0} = y_0(x), \tag{31}$$

where f is an inhomogeneity function, p is a pressure.

Auxiliary extremal problem:

$$\frac{\partial y}{\partial t} - \nu\Delta y + \sum_{i=1}^{3} u_i \frac{\partial y}{\partial x_i} = f(t,x) - \nabla p. \tag{32}$$

Here the function $u : (0,T)\times\Omega \to \mathrm{R}^3$ belongs to the class of solutions of (28)–(31) W. For each $u \in W$, problems (29)–(32) have a unique generalized solution. On solutions of (32), the following extremal problem is posed

$$J(u) = \|u - y(u)\|_W^2 \to \inf. \tag{33}$$

Under natural conditions, existence of a solution of (29)–(33) is proved using methods of optimal control theory. If for some u $J(u) = 0 \Leftrightarrow u = y$, there exists a solution for (28)–(31). In Chap. 6 we examine pullback attractors for a class of extremal solutions of the 3D Navier–Stokes system.

Example 4. We now consider a climate energy balance model proposed in [15, 33], and which has been studied from the dynamical point of view in several works (see. e.g., [70–72]). The problem is the following:

$$\begin{aligned}
&\tfrac{\partial u}{\partial t} - \tfrac{\partial^2 u}{\partial x^2} + Bu \in QS(x)\beta(u) + h(x), && (t,x) \in \mathbf{R}_+ \times (-1,1),\\
&u_x(-1,t) = u_x(1,t) = 0, && t \in \mathbf{R}_+,\\
&u(x,0) = u_0(x), && x \in (-1,1),
\end{aligned} \tag{34}$$

where B and Q are positive constants, $S, h \in L_\infty(-1,1)$, $u_0 \in L_2(-1,1)$, and β is a maximal monotone graph in $\mathbf{R}^2$, which is bounded, that is, there exist $m, M \in \mathbf{R}$ such that

$$m \le z \le M, \quad \text{for all } z \in \beta(s), s \in \mathbf{R}. \tag{35}$$

We also assume that

$$0 < S_0 \le S(x) \le S_1, \quad \text{a.e. } x \in (-1, 1). \tag{36}$$

The unknown $u(t, x)$ represents the averaged temperature of the Earth surface, Q is the so-called solar constant, which is the average (over a year and over the surface of the Earth) value of the incoming solar radiative flux, and the function $S(x)$ is the insolation function given by the distribution of incident solar radiation at the top of the atmosphere. When the averaging time is of the order of 1 year or longer, the function $S(x)$ satisfies (36), for shorter periods, we must assume that $S_0 = 0$. The term β represents the so-called co-albedo function, which can be possibly discontinuous. It represents the ratio between the absorbed solar energy and the incident solar energy at the point x on the Earth surface. Obviously, $\beta(u(x,t))$ depends on the nature of the Earth surface. For instance, it is well known that on ice sheets $\beta(u(x,t))$ is much smaller than on the ocean surface because the white color of the ice sheets reflects a large portion of the incident solar energy, whereas the ocean, due to its dark color and high heat capacity, is able to absorb a larger amount of the incident solar energy. We point out that this model is the particular case of (6.5). In fact, we cannot expect to have uniqueness for problem (34) either [15].

Example 5. Mathematical modelling of chemotaxis (the movement of biological cells or organisms in response to chemical gradients) has developed into a large and diverse discipline, whose aspects include its mechanistic basis, the modelling of specific systems, and the mathematical behavior of the underlying equations. From microscopic bacteria through to the largest mammals, the survival of many organisms is dependent on their ability to navigate within a complex environment through the detection, integration, and processing of a variety of internal and external signals. This movement is crucial for many aspects of behavior, including the location of food sources, avoidance of predators and attracting mates. The ability to migrate in response to external signals is shared by many cell populations, playing a fundamental role coordinating cell migration during organogenesis in embrionic development and tissue homeostasis in the adult. An acquired ability of cancer cells to migrate is believed to be a critical transitional step in the path to tumor malignancy [95].

The directed movement of cells and organisms in response to chemical gradients, *chemotaxis*, has attracted significant interest due to its critical role in a wide range of biological phenomena. In multicellular organisms, chemotaxis of cell populations plays a crucial role throughout the life cycle.

Theoretical and mathematical modelling of chemotaxis dates to the pioneering works of Patlak in the 1950s [211] and Keller and Segel in the 1970s [139, 140]. The review article by Horsmann [96] provides a detailed introduction into the mathematics of the Keller–Segel (KS) model for chemotaxis. This model can be summarized in the following (general) form [95]:

$$
\begin{aligned}
u_t &= \nabla(D(u)\nabla u - A(u)B(v)C(\nabla v)) + f(u), \\
v_t &= \triangle v + ug(u) - v,
\end{aligned}
\tag{37}
$$

on the domain $\Omega \in \mathrm{R}^n$ with prescribed initial data. Unless stated otherwise, we shall assume zero-flux boundary conditions:

$$
n \cdot (D(u)\nabla u - A(u)B(v)C(\nabla v)) = n \cdot \nabla v = 0,
$$

where n is the outer unit normal to $\partial\Omega$ and $\partial\Omega$ is piecewise smooth. In [95], it is assumed that the chemical signal acts as an *auto-attractant* and thus the chemical kinetics consist of cell-dependent chemical production and linear degradation. Where applicable, cell proliferation/death is assumed to be independent of the chemical signal. Due to the specific functional choices for $D(u)$, $A(u)$, $B(v)$, $C(\nabla v)$, $f(u)$, and $g(u)$ are given in the table below. In the majority of the models, $C(\nabla v)$ is simply given by ∇v, and we can define the *chemotaxis potential* $\phi(v)$ [23] to be the antiderivative of $B(v)$ such that

$$
B(v)\nabla v = \nabla\phi(v).
$$

Model	$D(u)$	$A(u)$	$B(v)$	$C(\nabla v)$	$f(u)$	$g(u)$
$(M1)$	D	u	χ	∇v	0	1
$(M2a)$	D	u	$\frac{\chi}{(1+\alpha v)^2}$	∇v	0	1
$(M2b)$	D	u	$\frac{\chi(\beta+1)}{(\beta+v)}$	∇v	0	1
$(M3a)$	D	$u(1-\frac{u}{\gamma})$	χ	∇v	0	1
$(M3b)$	D	$\frac{u}{1+\varepsilon u}$	χ	∇v	0	1
$(M4)$	D	u	χ	$\overset{\circ}{\nabla}_\rho v$	0	1
$(M5)$	Du^n	u	χ	∇v	0	1
$(M6)$	D	u	χ	∇v	0	$\frac{1}{1+\phi u}$
$(M7)$	D	u	χ	$\frac{1}{c}\tanh(\frac{c\nabla v}{1+c})$	0	1
$(M8)$	D	u	χ	∇v	$ru(1-u)$	1

where $(M1)$ is the minimal model

$$
\begin{aligned}
u_t &= \nabla(D\nabla u - \chi u \nabla v), \\
v_t &= \nabla^2 v + u - v.
\end{aligned}
\tag{38}
$$

$(M2a)$ is one of versions of signal-dependent sensitivity models, the "receptor" model,

$$
\begin{aligned}
u_t &= \nabla\left(D\nabla u - \frac{\chi u}{(1+\alpha v)^2}\nabla v\right), \\
v_t &= \nabla^2 v + u - v,
\end{aligned}
\tag{39}
$$

where for $\alpha \to 0$ the minimal model is obtained, and the "logistic" model

$$\begin{aligned} u_t &= \nabla\left(D\nabla u - \chi u \tfrac{1+\beta}{v+\beta}\nabla v\right), \\ v_t &= \nabla^2 v + u - v, \end{aligned} \tag{40}$$

where for $\beta \to \infty$ the minimal model follows and, for $\beta \to 0$, we obtain the classical form of $\chi(v) = 1/v$.

$(M3a)$ is one of density-dependent sensitivity models, the "volume-filling" model,

$$\begin{aligned} u_t &= \nabla\left(D\nabla u - \chi u(1 - \tfrac{u}{\gamma})\nabla v\right), \\ v_t &= \nabla^2 v + u - v, \end{aligned} \tag{41}$$

where the limit of $\gamma \to \infty$ leads to the minimal model, and

$$\begin{aligned} u_t &= \nabla\left(D\nabla u - \chi \tfrac{u}{1+\varepsilon u}\nabla v\right), \\ v_t &= \nabla^2 v + u - v, \end{aligned} \tag{42}$$

where $\varepsilon \to 0$ leads to the minimal model.

$(M4)$ is the nonlocal model

$$\begin{aligned} u_t &= \nabla\left(D\nabla u - \chi u \overset{\circ}{\nabla}_\rho v\right), \\ v_t &= \nabla^2 v + u - v, \end{aligned} \tag{43}$$

the nonlocal gradient $\overset{\circ}{\nabla}_\rho v$ is defined in Sect. 2.4 of [95] and chosen such that the minimal model follows for $\rho \to 0$.

$(M5)$ is the nonlinear-diffusion model

$$\begin{aligned} u_t &= \nabla\left(D\nabla u^n \nabla u - \chi u \nabla v\right), \\ v_t &= \nabla^2 v + u - v, \end{aligned} \tag{44}$$

where the minimal model corresponds to the limit of $n \to 0$.

$(M6)$ is the nonlinear signal kinetics model

$$\begin{aligned} u_t &= \nabla\left(D\nabla u - \chi u \nabla v\right), \\ v_t &= \nabla^2 v + \tfrac{u}{1+\phi u} - v, \end{aligned} \tag{45}$$

which approximates the minimal model for $\phi \to 0$.

$(M7)$ is the nonlinear gradient model

$$u_t = \nabla \left(D\nabla u - \chi u F_c(\nabla v) \right),$$
$$v_t = \nabla^2 v + u - v, \tag{46}$$

the vector-valued function F_c is defined in Sect. 2.7 of [95] and chosen such that the minimal model follows for $c \to 0$.

$(M8)$ is the cell kinetics model

$$u_t = \nabla \left(D\nabla u - \chi u \nabla v \right) + ru(1-u),$$
$$v_t = \nabla^2 v + u - v, \tag{47}$$

which in the limit of zero growth, $r \to 0$, leads to the minimal model.

If properly interpreting the derivative and correspondingly choosing phase spaces, all models can be reduced to autonomous first-order differential-operator equation with pseudomonotone type mappings. In this book, we are also present the results on asymptotical behavior of solutions for such models (see Chap. 2).

Another one application is the Ginzburg–Landau and Lotka–Volterra equation (see Chaps. 4 and 5).

Other important reason comes from the fact that there exists a gap between the conditions which are necessary to obtain the existence of solutions and the conditions that have to be imposed to prove the uniqueness. Therefore, avoiding uniqueness, we can weaken the conditions imposed on a differential equation and consider more general situation.

In this book, we deal with autonomous and nonautonomous multivalued dynamical systems, which are usually generated by an evolution equation or inclusion of the type

$$\frac{du}{dt} + A\left(t, u\left(t\right)\right) = \bar{0}, \; u\left(0\right) = u_0, \tag{48}$$

where for $t > 0$ $A(t, \cdot)$ is a function from reflexive Banach space V into its dual space V^*, or by a differential inclusion

$$\frac{du}{dt} + A\left(t, u\left(t\right)\right) \ni \bar{0}, \; u\left(0\right) = u_0, \tag{49}$$

where A is a multivalued map from reflexive Banach space V into its into nonempty convex closed bounded subsets of V^*, $(V; H, V^*)$ is an evolution triple [253].

When problem (48) (or (49)) is autonomous one and it possesses a unique solution for every initial data u_0, we can define a semigroups of operators $S : \mathbf{R}_+ \times H \to H$ by the rule $S\left(t, u_0\right) = u\left(t, u_0\right)$, where $u\left(t, u_0\right)$ denotes the unique solution corresponding to $u_0 \in H$. This operator satisfies the following properties:

$$S\left(0, u_0\right) = u_0, \;\; S\left(t+s, u_0\right) = S\left(t, S\left(s, u_0\right)\right), \;\; \forall\, t, s \geq 0.$$

The long-time behavior of the semigroup S can be described by the global attractor, which is an invariant set, usually compact and minimal, attracting uniformly every bounded set of H. The global attractor, when it exists, is a very important object for the understanding of the dynamics of a semigroup, as it contains the dynamics which is permanent with respect to time. At first, a theory of global attractors for infinite-dimensional dynamical systems was developed in [91, 155], where the results were applied to retarded ordinary differential equations and the two-dimensional Navier–Stokes system, respectively. It is worth to point out that the global attractor can help to explain turbulence in fluids and the mechanisms causing chaos (see [156,231]), and this is one of the reasons why this theory became popular very quickly. After that, this theory has been applied to a great number of equations like reaction-diffusion systems, wave equations, Kuramoto–Sivashinsky equations, phase-field equations, and many others.

However, if uniqueness in (48) (or (49)) does not hold, then a semigroup cannot be defined. Nevertheless, we can study the asymptotic behavior of solutions with the help of a multivalued semiflow given by

$$G\left(t, u_0\right)=\left\{\begin{array}{l} u\left(t\right) \mid u\left(\cdot\right) \text{ is a solution to (48)} \\ \text{of a certain class such that } u\left(0\right)=u_0 \end{array}\right\}.$$

This map has to satisfy the properties:

$$G\left(0, u_0\right)=u_0,$$

$$G\left(t+s, u_0\right) \subset G\left(t, G\left(s, u_0\right)\right), \forall t, s \geq 0.$$

If, moreover, $G(t+s, u_0)=G(t, G(s, u_0))$, then the multivalued semiflow is called strict, but this property does not always hold in applications. The main difference in the definition of the global attractor with respect to the single-valued case is that now the attractor has to be a negatively invariant set for the semiflow. Positively, semi-invariance is also obtained in many applications, but this is not always possible to prove.

We observe that multivalued semiflows have been used by several authors from time ago (see, e.g., [18, 30, 34, 143, 214, 215, 229]). However, the application to the study of global attractors is more recent (see [13, 14, 112, 119, 168, 176]). Chapters 1, 4 and 5 contain the general theory of attractors for multivalued semiflows as developed in [112, 113, 119, 168, 170, 176]. We give general sufficient conditions ensuring the existence of a global attractor and study additional topological properties as compactness, stability, and connectedness. Also, a theorem on continuous dependence of the attractor on a parameter is proved, and estimates of the fractal and Hausdorff dimensions are obtained. We observe that the results of the cited papers are now completed and improved. For example, the characterization of the global attractor as the union of all complete bounded trajectories or the result on stability is new. Also, the conditions leading to the estimates of the dimension are weakened.

There exist in the literature other methods to deal with the problem of non-uniqueness. On the one hand, we have the method of generalized semiflows, which has been developed in [16, 17, 76] (see [39] for a comparison of the two methods). On the other hand the theory of trajectory attractors has been also fruitfully applied to treat equations without uniqueness (see [51–53, 165, 191, 219]).

These theories have been applied successfully to a huge number of applications as differential inclusions (see [2,48,93,94,111,115,141,176,213,218,234,237,238]), reaction-diffusion equations (see [51, 75, 102, 109, 114, 116, 118, 185]), phase-field equations (see [108, 119, 184]), wave equations (see [16, 53]), the three-dimensional Navier–Stokes equations (see [16, 24, 53, 55, 117, 165, 219]), the three-dimensional Boussinesq equations (see [121, 191]), delay ordinary differential equations (see [42, 44]), lattice multivalued dynamical systems [184], or degenerate parabolic equations [76].

Although this book is mainly devoted to the theory of multivalued autonomous dynamical systems, it is interesting to point out that much attention has been paid over the last years to nonautonomous and stochastic dynamical systems in both the single-valued case (see, e.g., [22, 35, 36, 40, 43, 49, 50, 53, 59, 64–66, 79, 157, 217]) and the multivalued case (see, e.g., [37, 38, 41, 45, 110, 111, 119, 120, 177]).

This book consists of two parts. At first part we consider autonomous evolution inclusions with multivalued pseudomonotone type maps. The first chapter devoted to the abstract theory of multivalued semiflows, generated by several classes of evolution mathematical models and control problems with possibly nonlinear discontinuous or multivalued dependence between determinative parameters of the problem. At the second chapter, we consider new estimates and functional-topological properties of resolving operator of autonomous first- and second-order evolution inclusions with—coercive mappings. At the third chapter, compiling previous results, we investigate the long-time behavior of all weak solutions for abovementioned problems with several applications, involving Examples 1–4. We also consider at this part attractors for lattice dynamical systems with applications to "reaction-diffusion" type problems (see Chap. 4). The second part devoted to nonautonomous problems. We consider different approaches of examination of long-time behavior of all weak solutions for non-autonomous evolution inclusions with—coercive pseudomonotone type mappings with different applications like reaction-diffusion type systems (Chap. 4), Ginzburg–Landau equation and Lotka–Volterra system (Chap. 5), and 3D-Navier–Stokes equation (Chap. 6). We also consider new theorems on existence solutions and functional-topological properties of resolving operators with corresponding geophysical applications (Chap. 7).

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Chapter 1
Abstract Theory of Multivalued Semiflows

Beginning from the pioneering works [39, 52], the theory of global attractors of infinite-dimensional dynamical systems has become one of the main objects for investigation. Since then, deep results about existence, properties, structure, and dimension of global attractors for a wide class of dissipative systems have been obtained (see, e.g., [7, 38, 54, 75, 78]). For the application of this classical theory to partial and functional differential equations, it was necessary to have global existence and uniqueness of solutions of the Cauchy problem for all initial data of the phase space.

However, in many situations concerning systems of physical relevance, either uniqueness fails or it is not known to hold. In such cases, we cannot define a classical semigroup of operators, so that another theory involving multivalued maps is necessary: the theory of multivalued dynamical systems. It should be noticed that sometimes, the system is really multivalued (example of this cases will be given) but, in other cases, we just are not able to prove the uniqueness of solutions. Hence, the theory of multivalued dynamical systems allows us not to stop when the proof of uniqueness fails due to technical problems, and then, it is possible to study the asymptotic behavior of solutions no matter we have uniqueness or not. We observe that there is usually a gap between the conditions that we need to impose to obtain existence of solutions and the conditions necessary to prove uniqueness. Hence, we can weaken the conditions imposed on a differential equations and then consider more general situations.

Thus, there are infinite-dimensional problems whose qualitative behavior cannot be described by the methods mentioned above. First of all, we should mention the 3D Navier–Stokes system, for which it is not known whether uniqueness of weak solutions holds or not. On the other hand, although strong solutions are unique, it is not known whether such solutions exist globally in time or not (global regularity problem) [35, 74, 78]. A second class of problems of this type consists of nonlinear partial differential equations with non-Lipshitz or nonmonotone nonlinear terms, including reaction-diffusion systems (e.g., the complex Ginzburg–Landau equation or the Lotka–Volterra system), wave equations, or phase-field equations. For such

M.Z. Zgurovsky et al., *Evolution Inclusions and Variation Inequalities for Earth Data Processing III*, Advances in Mechanics and Mathematics 27,
DOI 10.1007/978-3-642-28512-7_1,

problems, uniqueness of solution does not hold in general (see, e.g., [10, 22, 26, 46, 47, 49, 51, 71]). Another class naturally appears when we deal with the wide class of problems described by evolution inclusions (which include control problems, several boundary-value problems or variational inequalities). In such problems the fact that the equation has a multivalued right-hand part usually guarantees the non-uniqueness of the Cauchy problem (see, e.g., [1, 2, 31, 68–71, 79, 80, 84, 85]). Of course, there exist much more examples of systems for which uniqueness can fail or it is not known to be true. Among them, we can cite the Euler equation [13, 29, 76], degenerate parabolic equations [32], delay ordinary differential equations with continuous nonlinear term [20, 37], some kinds of three-dimensional Cahn–Hilliard equations [72,73], the Boussinesq system [14,67], or lattice dynamical systems [66].

In order to treat such problems (with possible nonuniqueness of Cauchy problem) at the beginning of 1990s, three methods were proposed:

1. The method of multivalued semiflows.
2. The method of generalized semiflows.
3. The theory of trajectory attractors.

The method of multivalued semiflows [57, 63] (see also [4, 5] and the book [21]) and the method of generalized semiflows [9] were in fact very close and used the same idea: to allow nonuniqueness of the Cauchy problem and to consider the set (or some subset) of its solutions at every moment of time t. Hence, a multivalued analogous of a classical semigroup is considered. The main difference between them is that in the method of multivalued semiflows, it is considered a multivalued map from the phase space X onto a nonempty subset of the phase space for every moment of time (satisfying some properties similar to the classical ones for semigroups), whereas in the other method, the generalized semiflow is defined as a set of solutions satisfying some translation and concatenation properties, avoiding in this way multivalued maps. A comparison between these two theories can be found in [19]. These approaches appeared to be very useful and productive and allowed to obtain results about existence and properties of attractors for a wide class of dissipative systems without uniqueness (see, among others, [9, 10, 18, 20, 27, 44–48, 51, 63, 65, 71–73, 83, 84, 87]). The main difficulty of the method of multivalued semiflows is that we have to work with multivalued maps. Of course, in the case of uniqueness of solutions, all results in the two methods coincide with the classical ones. It is important to point out that this method for treating nonuniqueness, is in fact very old, as it was born many years before the theory of attractors for infinite-dimensional dynamical systems began to be developed in the 1970s. We can find multivalued semiflows already in the old paper [11] (see [8] for more references on the history of multivalued semiflows).

The third method is based on the idea of replacing the phase space of the problem by a specially constructed space of trajectories and considering on it the single-valued shift semigroup [22, 23, 55, 74]. The notion of trajectory attractor, that is, the global attractor (defined in the classical way) of the shift semigroup, also appeared to be very useful for the investigation of nonlinear evolution equations without uniqueness. In particular, the existence of a trajectory attractor has been proved for the 3D Navier–Stokes system [23, 34, 74] (see also [30], where a combination of these approaches is used), and also for other equations (see, e.g., [22, 23, 41, 56, 64, 67] and also [26]). As pointed out in [9], the disadvantage of this method is that the direct connection with the natural phase space of the problem is lost.

In this chapter, by using the theory of set-valued analysis, a qualitative research on multivalued dynamical systems is developed. First, we define multivalued flows (m-flow) and multivalued semiflows (m-semiflow) and study their ω-limit sets and global attractors. After that, we make a comparison between the method of multivalued semiflows and the method of trajectory attractors. These results were proved in the papers [50, 57, 63].

1.1 ω-Limit Sets and Global Attractors of Multivalued Semiflows

Let X be a complete metric space with metric $\rho(\cdot,\cdot)$; let $cl_X A$ denote the closure of A in X, $\mathbf{R}_+ = [0,\infty)$, Γ be a nontrivial subgroup of the additive group of the real numbers $\mathbf{R}$, $\Gamma_+ = \Gamma \cap \mathbf{R}_+$ and 2^X ($P(X)$, $\mathscr{B}(X)$, $C(X)$, $K(X)$) be the set of all (nonempty, nonempty bounded, nonempty closed, nonempty compact, respectively) subsets of X. For the multivalued map (m-map) $F : X \to 2^X$, we shall denote $D(F) = \{x \in X \mid F(x) \in P(X)\}$.

Definition 1.1. The m-map $G : \Gamma \times X \to P(X)$ is called a *multivalued flow (m-flow)* if the next conditions are satisfied:

1. $G(0,\cdot) = I$ is the identity map.
2. $G(t_1 + t_2, x) \subset G(t_1, G(t_2, x))$, $\forall t_1, t_2 \in \Gamma$, $\forall x \in X$,

where $G(t,B) = \underset{x\in B}{\cup} G(t,x)$, $B \subset X$.

The multivalued semiflow G is called *strict* if, moreover, $G(t+s,x) = G(t, G(s,x))$, $\forall t, s \in \mathbf{R}_+$, $\forall x \in X$.

Remark 1.1. The m-map $G : \Gamma_+ \times X \to P(X)$ is called an *m-semiflow* if conditions 1–2 of Definition 1.1 hold for any $t_1, t_2 \in \Gamma_+$.

Definition 1.2. The map $x(\cdot) : \Gamma_+ \to X$ is said to be a *trajectory* of the m-semiflow G corresponding to the initial condition x_0 if $x(t+\tau) \in G(t, x(\tau))$, $\forall t, \tau \in \Gamma_+$, $x(0) = x_0$.

We shall denote by $\mathscr{D}(x_0)$ the set of all trajectories corresponding to x_0.

For $x \in X$, $A, B \subset X$ we set $dist(x,B) = \inf_{y\in B}\{\rho(x,y)\}$. The distance from A to B will be defined by $dist(A,B) = \sup_{x\in A}\{dist(x,B)\}$.

Definition 1.3. It is said that the set $A \subset X$ *attracts* the set B with the help of the m-semiflow G if $dist(G(t,B), A) \to 0$ as $t \to +\infty$. The set M is said to be attracting for G if it attracts each $B \in \mathscr{B}(X)$.

For each $M \subset X$, we shall denote $\gamma_t^+(M) = \underset{\tau \geq t}{\cup} G(\tau, M)$ and $\gamma^+(M) = \gamma_0^+(M)$. The set $\omega(M) = \underset{t \geq 0}{\cap} \overline{\gamma_t^+(M)}$ is called the omega-limit (ω-limit) set of M. For $\epsilon > 0$, $B \in \mathscr{B}(X)$, the set $O_\epsilon(B) = \{y \in X \mid dist(y, B) < \epsilon\}$ is an ϵ-neighborhood of B.

Lemma 1.1. *The set $\omega(M)$ consists of the limits of all converging sequences $\{\xi_n\}$, where $\xi_n \in G(t_n, M)$, $t_n \to \infty$.*

Proof. It is obvious that $\overline{\gamma_{t_1}^+(M)} \subset \overline{\gamma_{t_2}^+(M)}$ for $t_1 \geq t_2$. Then, from the definition of $\omega(M)$ and [3, p. 18], it follows that

$$\omega(M) = \lim_{t \to \infty} \overline{\gamma_t^+(M)} = \lim_{t \to \infty} \gamma_t^+(M).$$

□

Definition 1.4. The m-semiflow G is called *asymptotically upper semicompact* if $\forall B \in \mathscr{B}(X)$ such that for some $T(B) \in \Gamma_+$, $\gamma_{T(B)}^+(B) \in \mathscr{B}(X)$, any sequence $\xi_n \in G(t_n, B)$, $t_n \to \infty$, is precompact in X

Definition 1.5. The set A is said to be *negatively semiinvariant* if $A \subset G(t, A)$, $\forall t \in \Gamma_+$. It is called invariant if $A = G(t, A)$, $\forall t \in \Gamma_+$.

Lemma 1.2. *Let $M \in \mathscr{B}(X)$ be a negatively semiinvariant set with respect to the m-semiflow G, which has an attracting set Z. Then $M \subset \overline{Z}$.*

Proof. Let $O(Z)$ be an arbitrary neighborhood of Z in X. Then $G(t, M) \subset O(Z)$, $\forall t \geq T(M)$. Since M is negatively semiinvariant and $O(Z)$ is arbitrary, we have $M \subset \overline{Z}$. □

Theorem 1.1. *Let the m-semiflow G be asymptotically upper semicompact. Then, for any $B \in \mathscr{B}(X)$ such that for some $T(B) \in \Gamma_+$, $\gamma_{T(B)}^+(B) \in B(X)$, $\omega(B) \neq \emptyset$ and it is a compact set in X. If, moreover, $\forall t \in \Gamma_+$, $G(t, \cdot): X \to C(X)$ is upper semicontinuous, then $\omega(B)$ is negatively semiinvariant and the minimal closed set attracting B. Moreover, $\omega(B)$ is connected if it attracts some connected bounded set $B_1 \supset \omega(B)$ and $G(t, x)$ is connected $\forall t \geq t_0$, $\forall x \in X$.*

Proof. Let $B \in \mathscr{B}(X)$ be such that $\gamma_{T(B)}^+(B) \in B(X)$ for some $T(B)$. It follows from the definition that the set $\omega(B)$ is bounded and closed. Let us consider an arbitrary sequence $\xi_n \in G(t_n, B)$, where $t_n \to \infty$. This sequence is precompact in X, since G is asymptotically upper semicompact. Hence, in view of Lemma 1.1, $\omega(B) \neq \emptyset$.

Let us prove that $\omega(B)$ is compact. Let $\{y_n\}$ be a sequence belonging to $\omega(B)$. Then there exist $z_n \in G(\tau_n, B)$, $\tau_n \in \Gamma_+$, $\tau_n \to \infty$ as $n \to \infty$, such that $\|y_n - z_n\| < 1/n$. Since G is asymptotically upper semicompact, we can choose a subsequence z_m such that $z_m \to z_0$ in X, $z_0 \in \omega(B)$, and $\rho(y_m, z_m) \to 0$. Hence, there exists a converging subsequence $y_m \to z_0$, and then, $\omega(B)$ is compact.

Let us now prove that $\omega(B)$ is negatively semiinvariant. Let $\xi \in \omega(B)$. Then there exists a sequence $\xi_n \in G(t_n, B)$, such that $\xi_n \to \xi$ in X, as $t_n \to \infty$. For $t_n \geq t$, we have $G(t_n, B) \subset G(t, G(t_n - t, B))$ and, therefore, $\xi_n \in G(t, \zeta_n)$, where $\zeta_n \in G(t_n - t, B)$. Since the m-semiflow G is asymptotically upper semicompact and $t_n - t \to \infty$ as $n \to \infty$, there exists a subsequence $\{\zeta_{n_m}\}$ such that $\zeta_{n_m} \to \zeta \in \omega(B)$. Thus, without loss of generality, we can consider that $\xi_n \to \xi$ and $\zeta_n \to \zeta$ in X. Since $G(t, \cdot)$ is upper semicontinuous and has closed values, its graph is closed (see [3, p. 42]). Hence, $\xi \in G(t, \omega(B))$. Finally, $\omega(B) \subset G(t, \omega(B))$, $\forall t \in \Gamma_+$, because $\xi \in \omega(B)$ and $t \in \Gamma_+$ are arbitrary.

The set $\omega(B)$ attracts B. Indeed, let it not be so. Then we can find a sequence $\xi_m \in G(t_m, B)$, $t_m \to \infty$, such that $dist(\xi_m, \omega(B)) > \epsilon > 0$. On the other hand, since G is asymptotically upper semicompact, this sequence has at least one subsequence converging to some $\zeta \in \omega(B)$. The resulting contradiction proves the statement.

Let us prove the minimality of $\omega(B)$. Let F be a closed set which attracts B and $\omega(B) \not\subset F$. Then there exist $x \in \omega(B)$, $x \notin F$ and neighborhoods $O_\epsilon(x)$, $O_\epsilon(F)$ such that $O_\epsilon(x) \cap O_\epsilon(F) = \emptyset$. From Lemma 1.1, it follows that $x = \lim_{n\to\infty} y_n$, where $y_n \in G(t_n, x_n)$, $x_n \in B$. On the other hand, $y_n \in O_\epsilon(F)$, $\forall t_n \geq T(B)$, which is a contradiction. Hence, $\omega(B) \subset F$.

Finally, let us prove that $\omega(B)$ is connected if the conditions of the theorem hold. Let $\omega(B)$ attracts some bounded connected set $B_1 \supset \omega(B)$. Suppose the opposite, that is, $\omega(B)$ is not connected. Then we can find two disjoint compact sets Ω_1 and Ω_2 such that $\omega(B) = \Omega_1 \cup \Omega_2$. We take $\epsilon > 0$ so small that $O_\epsilon(\Omega_1) \cap O_\epsilon(\Omega_2) = \emptyset$. In such a case $O_\epsilon(\omega(B)) = O_\epsilon(\Omega_1) \cup O_\epsilon(\Omega_2)$ is an ϵ-neighborhood of the set $\omega(B)$. Since $\omega(B)$ attracts B_1, we can find $t_1 \in \Gamma_+$ such that $G(t, B_1) \subset O_\epsilon(\omega(B))$, $\forall t \geq t_1$. It is well known (see [15, 33]) that an upper semicontinuous map with connected values maps any connected set into a connected one. Then $G(t_1, B_1)$ is connected and belongs completely to one of the sets $O_\epsilon(\Omega_i)$, $i = 1, 2$. Since $\omega(B)$ is negatively semiinvariant, we have $\omega(B) \subset G(t_1, \omega(B)) \subset G(t_1, B_1)$, which is a contradiction. □

Remark 1.2. It follows from the proof of Theorem 1.1 that all the statements of the theorem (with the exception of the connectivity) hold if the map $G(t, \cdot) : X \to P(X)$ is not upper semicontinuous but has closed graph.

Remark 1.3. In the single-valued case, we have that any asymptotically upper semicompact m-semiflow is an asymptotically compact semigroup (see [54]).

The next propositions are useful to check in applications that an m-semiflow is asymptotically upper semicompact.

Proposition 1.1. *Let the map $G(t, \cdot) : X \to P(X)$ be compact for some $t_1 \in \Gamma_+ \setminus \{0\}$, that is, $\forall B \in \mathscr{B}(X)$, $G(t_1, B)$ is precompact in X. Then the m-semiflow G is asymptotically upper semicompact.*

Proof. Let $\xi_n \in G(t_n, B)$ be an arbitrary sequence, where $t_n \to \infty$. Let $B \in \mathscr{B}(X)$ be such that there exists $T(B) \in \Gamma_+$ for which $\gamma^+_{T(B)}(B) \in \mathscr{B}(X)$. Obviously,

$\gamma^+_{t_1}(B) \subset \gamma^+_{t_2}(B)$, $\forall t_1 \geq t_2$. Hence, $\gamma^+_\tau(B) \in \mathscr{B}(X)$, if $\tau \geq T(B)$. Further, by using the properties of m-semiflows, we get

$$\gamma^+_{t_1+\tau}(B) = \bigcup_{s\geq\tau} G(t_1+s, B) \subset \bigcup_{s\geq\tau} G(t_1, G(s,B)) = G\left(t_1, \gamma^+_\tau(B)\right)$$

Since $\gamma^+_\tau(B) \in \mathscr{B}(X)$, $G(t_1, \gamma^+_\tau(B))$ is precompact and then

$$\gamma^+_{t_1+\tau}(B) \subset G(t_1, \gamma^+_\tau(B))$$

is also precompact. Consequently, since for n such that $t_n \geq \tau + t_1$, $\xi_n \in \gamma^+_{\tau+t_1}(B)$, one deduces that from $\{\xi_n\}$, we can extract a converging subsequence. □

Lemma 1.3. *Let $G(t,\cdot) : X \to K(X)$ be upper semicontinuous $\forall t \in \Gamma_+$ and let for some $t_1 \in \Gamma_+ \setminus \{0\}$ the map $G(t_1,\cdot)$ be compact. Then the map $G(t_1+t,\cdot)$ is also compact for any $t \in \Gamma_+$.*

Proof. Let $B \in \mathscr{B}(X)$. Obviously $G(t_1+t, B) \subset G(t, G(t_1, B)) \subset G(t, \overline{G(t_1, B)})$. Since $G(t,\cdot)$ is upper semicontinuous and has compact values, the set $G(t, \overline{G(t_1, B)})$ is compact in X (see [2, p. 42]). Thus the set $G(t+t_1, B))$ is precompact. □

Proposition 1.2. *Let X be a Banach space with norm $\|\cdot\|$. Let $G(t,\cdot) = \Gamma(t,\cdot) + K(t,\cdot)$ be an m-semiflow, where $K(t_0,\cdot) : X \to P(X)$ is a compact map for some $t_0 \in \Gamma_+ \setminus \{0\}$ and $\Gamma(t,\cdot) : X \to P(X)$ is a contraction on bounded sets, that is,*

$$dist(\Gamma(t,x), \Gamma(t,y)) \leq m_1(t)\, m_2(\|x-y\|), \quad \forall x, y \in B \in \mathscr{B}(X), \quad \forall t \in \Gamma_+, \tag{1.1}$$

where $m_2 : \mathbf{R}_+ \to \mathbf{R}$ is a continuous map and $m_1 : \Gamma_+ \to \mathbf{R}$ is a decreasing map such that $m_1(t) \to 0$, as $t \to \infty$. Then G is asymptotically upper semicompact.

Proof. Let $B \in \mathscr{B}(X)$ be such that $\exists T(B) \in \Gamma_+$ for which $\gamma^+_{T(B)}(B) \in \mathscr{B}(X)$. Let us consider an arbitrary sequence $N = \{\xi_n\}_{n=1}^\infty$, where $\xi_n \in G(t_n, B)$, $t_n \to \infty$, and prove that N is completely bounded. For a fixed $\epsilon > 0$, we take $t_1 = t_1(\epsilon) \geq t_0$ such that

$$m_1(t_1) \leq \epsilon \left(2m_2\left(diam\, \gamma^+_{T(B)}(B)\right)\right)^{-1}$$

and decompose N into two sets $N = N_1 \cup N_2$, where $N_1 = \{\xi_n\}_{n=1}^{n_1}$, $t_n < t_1 + T(B)$, $N_2 = \{\xi_n\}_{n_1+1}^\infty$, $t_n \geq t_1 + T(B)$. Since $\gamma^+_{t_1+T(B)}(B) = \bigcup_{t\geq T(B)} G(t+t_1, B) \subset \bigcup_{t\geq T(B)} G(t_1, G(t,B)) = G(t_1, \gamma^+_{T(B)}(B))$, we have $N_2 \subset G(t_1, \gamma^+_{T(B)}(B))$. On the other hand,

$$\gamma^+_{t_1+T(B)}(B) = \bigcup_{t\geq T(B)+t_1-t_0} G(t+t_0, B) \subset \bigcup_{t\geq T(B)+t_1-t_0} G(t_0, G(t,B))$$

$$= G(t_0, \gamma^+_{T(B)+t_1-t_0}(B)) \subset G(t_0, \gamma^+_{T(B)}(B)),$$

so that $N_2 \subset G(t_0, \gamma^+_{T(B)}(B))$ as well. Since $K\left(t_0, \gamma^+_{T(B)}(B)\right)$ is precompact and in view of (1.1), $diam\Gamma\left(t_1, \gamma^+_{T(B)}(B)\right) \leq \frac{\varepsilon}{2}$, we can find a finite ε-net for $\gamma^+_{t_1+T(B)}(B)$. So, N is completely bounded and by Hausdorff's theorem it is precompact. □

Corollary 1.1. *Let X be a complete metric space and let the m-semiflow $G(t, \cdot) : X \to P(X)$ satisfy (1.1) but replacing $\|x - y\|$ by $\rho(x, y)$. Then, G is asymptotically upper semicompact.*

Definition 1.6. The set $\Re$ is called a global attractor of the m-semiflow G if it satisfies the next conditions:

1. $\Re$ attracts any $B \in \mathscr{B}(X)$.
2. $\Re$ is negatively semiinvariant, that is, $\Re \subset G(t, \Re)$, $\forall t \in \Gamma_+$.

Remark 1.4. In [57, 62], the global attractor was supposed to be closed and bounded. We have removed these properties because there are systems for which the minimal global attractor can be unbounded.

Remark 1.5. It follows from the preceding definition that the next properties hold:

1. For any bounded negatively semiinvariant set A, $A \subset cl_X(\Re)$.
2. If $\Re$ is bounded, for any closed set Z attracting each $B \in \mathscr{B}(X)$, we have $\Re \subset Z$. Hence, if $\Re$ is closed, it is minimal among all closed sets attracting each $B \in \mathscr{B}(X)$.

Proof. Let $M \in \mathscr{B}(X)$ be negatively semiinvariant. For any ε-neighborhood $O_\varepsilon(\Re))$ of $\Re$, there exists T such that $G(t, M) \subset O_\varepsilon(\Re)$, $\forall t \geq T$. Then, $M \subset G(t, M) \subset O_\varepsilon(\Re)$. Being $O_\varepsilon(\Re)$ arbitrary, $M \subset cl_X(\Re)$. The second statement follows from Lemma 1.2. □

Definition 1.7. The m-semiflow G is called pointwise dissipative if $\exists B_0 \in \mathscr{B}(X)$ attracting any point $x \in X$.

Theorem 1.2. *Let G be asymptotically upper semicompact m-semiflow. Suppose that $G(t, \cdot) : X \to C(X)$ is upper semicontinuous for any $t \in \Gamma_+$. If $\forall B \in \mathscr{B}(X)$ $\exists T(B) \in \Gamma_+$ such that $\gamma^+_{T(B)}(B) \in \mathscr{B}(X)$, then G has the global attractor $\Re$ defined by*

$$\Re = \bigcup_{B \in \mathscr{B}(X)} \omega(B),$$

which is locally compact in the sum topology $\tau_\oplus$. Moreover, for any closed set Z attracting each $B \in \mathscr{B}(X)$, $\Re \subset Z$. The space $(\Re, \tau_\oplus)$ is Lindelöf.

Proof. It follows from Theorem 1.1 that $\Re$ attracts each $B \in \mathscr{B}(X)$. It follows also that for any closed set Z attracting each bounded set and any $B \in \mathscr{B}(X)$, $\omega(B) \subset Z$. Hence, $\Re \subset Z$. Since each $\omega(B)$ is negatively semiinvariant, the set $\Re$ is also the same one. Indeed,

$$\mathfrak{R} \subset \bigcup_{B \in \mathscr{B}(X)} G(t, \omega(B)) \subset G(t, \mathfrak{R})$$

Since $\omega(B_1) \subset \omega(B_2)$ if $B_1 \subset B_2$, it is clear that $\mathfrak{R} \subset \cup_{i=1}^{\infty}\omega(B_i)$, where $B_i = \{x \in X \mid \|x\| \le i\}$. On the other hand, $\cup_{i=1}^{\infty}\omega(B_i) \subset \mathfrak{R}$. Hence, $\cup_{i=1}^{\infty}\omega(B_i) = \mathfrak{R}$. Each $\omega(B_i)$ is homeomorphic to the space $D_i = \{(x,i) \mid x \in \omega(B_i)\}$. Hence, accurate to homeomorphisms $\mathfrak{R} = \cup_{i=1}^{\infty} D_i$, where D_i are compact and $D_i \cap D_j = \emptyset$, if $i \neq j$ (see [86, p. 65]). Each D_i is a topological space with the topology τ_i induced by X. We consider the family $\beta_{\oplus} = \{U \subset \mathfrak{R} \mid U \cap D_i \in \tau_i$ for any $i \ge 1\}$, which is a sub-base of a topology $\tau_{\oplus}$ in $\mathfrak{R}$. $\tau_{\oplus}$ is called the sum topology. Each set D_i is open and closed in $(\mathfrak{R}, \tau_{\oplus})$. Hence, $(\mathfrak{R}, \tau_{\oplus})$ is locally compact. Indeed, let $x \in \mathfrak{R}$. Then $x \in D_i$, so D_i is a neighborhood of x in the topology $\tau_{\oplus}$. Since D_i is regular, we can find some open set $U(x) \in \tau_{\oplus}$ such that $\overline{U(x)} \subset D_i$. $\overline{U(x)}$ is a compact neighborhood of x in $(\mathfrak{R}, \tau_{\oplus})$. Moreover, since $\mathfrak{R}$ is the countable union of compact sets, $(\mathfrak{R}, \tau_{\oplus})$ is Lindelöf. It follows also that $(\mathfrak{R}, \tau_{\oplus})$ is a normal space. □

Remark 1.6. Let X be an infinite-dimensional Banach space. Then since

$$\mathfrak{R} = \bigcup_{i=1}^{\infty} \omega(B_i),$$

it follows from Baire's theorem that $\mathfrak{R} \neq X$. However, we note that $\mathfrak{R}$ can be dense in X. Theorem 1.2 remains valid if the map $G(t,\cdot) : X \to C(X)$ is not upper semicontinuous but has closed graph.

Consider now some theorems which state the existence of compact attractors for m-semiflows.

Theorem 1.3. *Let G be a pointwise dissipative and asymptotically upper semicompact m-semiflow. Suppose that $G(t,\cdot) : X \to C(X)$ is upper semicontinuous for any $t \in \Gamma_+$. If $\forall B \in \mathscr{B}(X)$, $\exists T(B) \in \Gamma_+$ such that $\gamma^+_{T(B)}(B) \in \mathscr{B}(X)$, then G has the compact global attractor $\mathfrak{R}$. It is minimal among all closed sets attracting each $B \in \mathscr{B}(X)$.*

Proof. In view of Theorem 1.1, $\forall B \in \mathscr{B}(X)$, the ω-limit set $\omega(B)$ is non-empty, compact, and negatively semiinvariant, and it is the minimal closed set attracting B. We set $B_1 = O_{\epsilon_1}(B_0)$, where $\epsilon_1 > 0$ is some fixed number and B_0 is the set which attracts each point $x \in X$. Let $\mathfrak{R} = \omega(B_1)$. Let us prove that $\mathfrak{R}$ is a global attractor. Let $B \in \mathscr{B}(X)$, $x \in \omega(B)$, be arbitrary. It is clear that there exists $T(x) \in \Gamma_+$ such that $G(T,x) \subset B_1$. On the other hand, since $G(t,\cdot) : X \to P(X)$ is upper semicontinuous, there exists an open neighborhood $O(x)$ such that $G(T(x), O(x)) \subset B_1$. Further, from the covering $\cup_{x\in\omega(B)} O(x)$ of the compact $\omega(B)$, we can extract a finite subcovering $O(\omega(B)) = \cup_{i=1}^{n(B)} O(x_i)$, $x_i \in \omega(B)$. For each point x_i, the next inclusions hold $G(t + T(x_i), O(x_i)) \subset G(t, G(T(x_i), O(x_i))) \subset G(t, B_1) \subset O_{\epsilon_2}(\omega(B_1))$, $\forall t \ge T(\epsilon_2, B_1)$. Hence,

$$G(t, O(\omega(B))) \subset O_{\epsilon_2}(\omega(B_1)), \ \forall t \geq T(\epsilon_2, B_1) + \widehat{t}$$

where $\widehat{t} = \max\{T(x_i)\}$. Further, $\forall \epsilon > 0$, $\exists T(\epsilon, B)$ such that $G(t, B) \subset O_\epsilon(\omega(B))$, $\forall t \geq T(\epsilon, B)$, and $O(\omega(B))$ contains some $\epsilon(B)$-neighborhood $O_{\epsilon(B)}(\omega(B))$ of the set $\omega(B)$. Then $G(t, B) \subset O_{\epsilon(B)}(\omega(B)) \subset O(\omega(B))$, $\forall t \geq T(\epsilon(B), B)$ and therefore $G(t, B) \subset O_{\epsilon_2}(\omega(B_1))$, $\forall t \geq T(\epsilon(B), B) + T(\epsilon_2, B_1) + \widehat{t}$. Hence, $\omega(B_1)$ attracts any $B \in \mathscr{B}(X)$. It follows from Theorem 1.1 that $\Re$ is compact and negatively semi-invariant. The minimality of $\Re$ follows from Remark 1.5. □

Theorem 1.4. *Let* $\forall t \in \Gamma_+$, $G(t, \cdot) : X \to C(X)$ *be an upper semicontinuous map. If there exists a compact set* $K \subset X$ *such that* $\forall B \in \mathscr{B}(X)$

$$dist(G(t, B), K) \to 0 \ \ as \ \ t \to \infty, \tag{1.2}$$

the m-semiflow G *has the global compact attractor* $\Re \subset K$. *It is the minimal closed set attracting each* $B \in \mathscr{B}(X)$.

Proof. Let $B \in \mathscr{B}(X)$. We shall prove that $\omega(B) \neq \emptyset$ and $\omega(B) \subset K$. Since $\omega(B) = \lim_{t\to\infty} \overline{\gamma_t^+(B)}$ (see Lemma 1.1), it follows from (1.2) that $\omega(B) \subset K$. It remains to prove that $\omega(B) \neq \emptyset$ if $B \neq \emptyset$. Let $t_n \to \infty$ and $\xi_n \in G(t_n, B)$. In view of (1.2), there exists a sequence $\zeta_n \in K$ such that $\rho(\xi_n, \zeta_n) \to 0$. Since K is compact, from the sequence $\{\zeta_n\}$, we can extract a converging subsequence $\zeta_{n_m} \to \zeta \in K$. Consequently, $\zeta \in \omega(B)$. By definition, the set $\omega(B)$ is closed and then compact. Let us prove that $\omega(B)$ is negatively semiinvariant. Let $\xi \in \omega(B)$. Then there exists $\xi_n \in G(t_n, B)$ such that $\xi_n \to \xi$ as $t_n \to \infty$. For $t_n \geq t$, we have $G(t_n, B) \subset G(t, G(t_n - t, B)$ and then $\xi_n \in G(t, \zeta_n)$, where $\zeta_n \in G(t_n - t, B)$. On the other hand $dist(G(t_n - t, B), K) \to 0$ as $t_n \to \infty$. Hence, there exists a sequence $\{a_n\}$, $a_n \in K$, such that $\rho(\zeta_n, a_n) \to 0$ as $n \to \infty$. Taking a subsequence if necessary, we have $\zeta_n \to \zeta$ and $a_n \to \zeta$ in X. Therefore, we can put $\xi_n \to \xi$, $\zeta_n \to \zeta$ and $\xi_n \in G(t, \zeta_n)$. Since the map $G(t, \cdot)$ is upper semicontinuous and has closed values, it is closed (see [3, p. 42]). Therefore, we have $\xi \in G(t, \zeta)$. Since ξ and $t \in \Gamma_+$ are arbitrary, we obtain $\omega(B) \subset G(t, \omega(B))$, $\forall t \in \Gamma_+$.

We set $\Re = \omega(K)$. This set is a global attractor of G. We must prove the attracting property. Assume the opposite, that is, for some $B \in \mathscr{B}(X)$, $\exists \epsilon > 0$ such that

$$\limsup_{t\to\infty} dist(G(t, B), \Re) = 2\epsilon > 0$$

In such a case, there exist $t_n \to \infty$ and $\xi_n \in G(t_n, B)$ such that $dist(\xi_n, \Re) \geq \epsilon$, $\forall t_n \geq T$. On the other hand, there is no loss of generality in assuming $\xi_n \to \xi$ in K. Hence, $\xi \in \omega(B) \subset \Re$. The resulting contradiction proves the attracting property. The minimality of $\Re$ follows from Remark 1.5. □

Remark 1.7. In Theorems 1.3, 1.4 if $G(t_1 + t_2, x) = G(t_1, G(t_2, x))$, $\forall x \in X$, $\forall t_1, t_2 \in \Gamma_+$, then $\Re = G(t, \Re)$, $\forall t \in \Gamma_+$.

Proof. We know that $\mathfrak{R} \subset G(t, \mathfrak{R})$, $\forall t \in \Gamma_+$. Let us prove the inverse. For any $t, \tau \in \Gamma_+$, we have

$$G(t, \mathfrak{R}) \subset G(t, G(\tau, \mathfrak{R})) \subset G(t + \tau, \mathfrak{R})$$

For any ε-neighborhood $O_\varepsilon(\mathfrak{R})$ of $\mathfrak{R}$, there exists T such that $G(t+\tau, \mathfrak{R}) \subset O_\varepsilon(\mathfrak{R})$, $\forall \tau \geq T$. Since $O_\varepsilon(\mathfrak{R})$ is arbitrary, we have $G(t, \mathfrak{R}) \subset cl_X(\mathfrak{R}) = \mathfrak{R}$. Hence, $\mathfrak{R} = G(t, \mathfrak{R}),\ \forall t \in \Gamma_+$. □

Theorem 1.4 remains valid if the map $G(t, \cdot) : X \to C(X)$ is not upper semicontinuous but has closed graph. Is Theorem 1.3 valid if the map $G(t, \cdot) : X \to C(X)$ is not upper semicontinuous but has closed graph? This statement is misleading, as in such a case we need an extra assumption, namely, that the semiflow G is strict.

We have:

Lemma 1.4. *The statement of Theorem 1.3 holds if $x \to G(t, x)$ is not upper semicontinuous but has closed graph and G is a strict semiflow.*

Proof. In view of Theorem 1.1 and Remark 1.1 for any $B \in \mathscr{B}(X)$, the omega-limit set $\omega(B)$ is nonempty, compact, and negatively semiinvariant and it is the minimal closed set attracting B.

Since G is pointwise dissipative, there exists a bounded set $\overline{B}$ attracting every $x \in X$. Let

$$B_0 = \{y \in X : dist\left(y, \overline{B}\right) < \varepsilon\},$$

where $\varepsilon > 0$ and $dist(\cdot, \cdot)$ is the Hausdorff semidistance. Then B_0 is pointwise absorbing, that is, for any $x \in X$, there exists $T(x)$ such that $G(t, x) \subset B_0$, for all $t \geq T$. We have to check that for every bounded set B, we have

$$\omega(B) \subset \omega(B_0).$$

From this, it follows easily that $\mathfrak{R} = \omega(B_0)$ is a global compact attractor, as $\omega(B_0)$ is negatively semiinvariant and attracts every $B \in \mathscr{B}(X)$. By Remark 1.5, we know also that $\omega(B_0)$ is minimal among all closed sets attracting each $B \in \mathscr{B}(X)$.

Suppose that $y_n \in G(t_n, x_n)$, $t_n \to \infty$, $x_n \in B$, $y_n \to y$, and $y \notin \omega(B_0)$. We shall obtain a contradiction. Since $G(t_n, x_n) \subset G\left(\frac{t_n}{2}, G\left(\frac{t_n}{2}, x_n\right)\right)$, there exist z_n such that $y_n \in G\left(\frac{t_n}{2}, z_n\right)$ and $z_n \in G\left(\frac{t_n}{2}, x_n\right)$. Passing to a subsequence we get

$$z_n \to z.$$

We can choose $T(z)$ such that

$$G(t, z) \subset B_0, \forall t \geq T.$$

Then we take ξ_n such that $y_n \in G\left(\frac{t_n}{2} - T, \xi_n\right)$, $\xi_n \in G(T, z_n)$. Since G is a strict semiflow, we have

$$\xi_n \in G\left(\frac{t_n}{2} + T, x_n\right),$$

and then (up to a subsequence)

$$\xi_n \to \xi.$$

But the graph of G is closed, so $\xi \in G\,(T, z)$.

Then $\xi \in B_0$ and $\xi_n \to \xi$ imply

$$\xi_n \in B_0, \forall n \geq n_0.$$

Hence, $y_n \to y$, $y_n \in G\left(\frac{t_n}{2} - T, B_0\right)$, so that

$$y \in \omega(B_0).$$

□

When the semiflow G is not strict, the result is also true if we assume that G has an absorbing bounded set B_0. This means that for any bounded set B, there exists $T\,(B)$ such that $G\,(t, B) \subset B_0$, for all $t \geq T$. This property is obviously stronger than being pointwise dissipative.

Lemma 1.5. *The statement of Theorem 1.3 holds if $x \to G\,(t, x)$ is not upper semicontinuous but has closed graph and G has a bounded absorbing set.*

Proof. In this case, the results follows easily by taking $\Re = \omega\,(B_0)$. Indeed, in view of Theorem 1.1 and Remark 1.1, the omega-limit set $\omega\,(B_0)$ is nonempty, compact, and negatively semiinvariant, and it is the minimal closed set attracting B_0. Also, $\omega\,(B_0)$ attracts every bounded set B as for any $\varepsilon > 0$, there exists $\overline{t}\,(\varepsilon, B)$ such that

$$G\,(t, B) \subset G\,(t - T\,(B)\,, G\,(T\,(B)\,, B)) \subset G\,(t - T\,(B)\,, B_0) \subset O_\varepsilon\,(\omega\,(B_0))\,,$$

if $t \geq \overline{t}\,(\varepsilon, B)$, where $O_\varepsilon\,(\omega\,(B_0))$ is an ε-neighborhood of $\omega\,(B_0)$. By Remark 1.5, we know also that $\omega\,(B_0)$ is minimal among all closed sets attracting each $B \in \mathscr{B}(X)$. □

Let us consider now the connectivity of the global attractor. We shall follow the method used in [36] for the single-valued case.

Definition 1.8. The m-semiflow $G : \mathbf{R}_+ \times X \to P(X)$ is said to be timecontinuous if it is the union of continuous trajectories, that is,

$$G(t, x_0) = \{x(t) \mid x(\cdot) \in \mathscr{D}(x_0), x(\cdot) \in C(\mathbf{R}_+, X)\}, \forall x_0 \in X$$

Theorem 1.5. *Let us suppose that the conditions of Theorems 1.3 or 1.4 are satisfied. Assume also that G is a time-continuous m-semiflow with connected values such that $G(t_1 + t_2) = G(t_1, G(t_2, x))$, $\forall t_1, t_2 \in \Gamma_+$, $\forall x \in X$. If the space X is connected, then the global attractor $\Re$ is connected.*

Proof. We suppose that the attractor $\Re$ is not connected. Then we can find two nonempty compact disjoint sets A_1 and A_2 such that $A_1 \cup A_2 = \Re$ and $O_\epsilon(A_1) \cap O_\epsilon(A_2) = \emptyset$, for some $\epsilon > 0$. Let us define the next disjoint sets

$$X_i = \{x \in X \mid G(t,x) \subset O_\epsilon(A_i), \forall t \geq T(x)\},\ i = 1,2.$$

If we prove that $X_1 \cup X_2 = X$ and that X_1 and X_2 are nonempty open sets, then we obtain a contradiction, since we have supposed X to be connected.

First, let us show that $X_1 \cup X_2 = X$. Let $x \in X$ be arbitrary. There exists $T \geq 0$ such that $G(t,x) \subset O_\epsilon(\Re)$, $\forall t \geq T$. Since G has connected values, $G(T,x)$ belongs to one of the sets $O_\epsilon(A_i)$. Consider the set $G([T,+\infty),x) = \cup_{t \geq T} G(t,x)$. It is clear that for each continuous trajectory $x(\cdot)$ corresponding to x, the set $x([T,+\infty)) = \cup_{t\geq T} x(t)$ is connected and then it is completely contained in $O_\epsilon(A_i)$. Hence, since G is time continuous, $G([T,+\infty),x) \subset O_\epsilon(A_i)$. Therefore, $X = X_1 \cup X_2$.

Next, we shall show that the sets X_i are nonempty. We shall prove that $\emptyset \neq A_i \subset X_i$, $i = 1,2$. Since $\Re$ is invariant (i.e., $G(t,\Re) = \Re$), using the same argument as before, we have that for any $x \in A_i$, $G([0,+\infty),x) \subset A_i$.

Finally, we shall prove that X_1 and X_2 are open sets. Let $x \in X_i$ and B be a bounded neighborhood of x. There exists $T \geq 0$ such that $G(t,B) \subset O_\epsilon(\Re)$, $\forall t \geq T$. As the map $G(T,\cdot)$ is upper semicontinuous, there exists a neighborhood $U \subset B$ of x such that $G(T,U) \subset O_\epsilon(A_i)$. Arguing as before for each $y \in U$, we get $G(t,U) \subset O_\epsilon(A_i)$, $\forall t \geq T$. Hence, $U \subset X_i$. Since x is arbitrary, X_i is open.

□

Remark 1.8. In [77], was proved the connectivity of the global attractor in the case when the multivalued map $G : \mathbf{R}_+ \times X \to 2^X$ is upper semicontinuous with respect to $\{t,x\} \in \mathbf{R}_+ \times X$. In [36], an example of a discontinuous attractor is given.

Theorem 1.6. *Let us suppose that the conditions of Theorems 1.3 or 1.4 are satisfied. Assume also that G has connected values and that*

$$\Re \subset B_1 \in \beta(X),$$

where B_1 is connected. Then the global attractor $\Re$ is connected.

Proof. Suppose that $\Re$ is not connected. Then there exist two open sets A_1, A_2 such that $\Re \cap A_1 \neq \emptyset$, $\Re \cap A_2 \neq \emptyset$, $\Re \subset A_1 \cup A_2$, and $A_1 \cap A_2 = \emptyset$.

Since the map $x \mapsto G(t,x)$ is upper semicontinuous and has connected values, $G(t,B_1)$ is a connected set. Indeed, if $G(t,B_1)$ were not connected, then there would exist open sets U_1 and U_2 with $U_1 \cap U_2 = \emptyset$ such that $G(t,B_1) \cap U_i \neq \emptyset$, $i = 1$, 2, and $G(t,B_1) \subset U_1 \cup U_2$. Denote $M_i = \{x \in B_1 : G(t,x) \subset U_i\}$. Since $G(t,x)$ has connected values, $M_1 \cup M_2 = B_1$. We can see that $M_1 \cap M_2 = \emptyset$ and $M_i \neq \emptyset$ for $i = 1,2$. Since $x \mapsto G(t,x)$ is upper semicontinuous, M_i are open sets for $i = 1$, 2 (see [42, p. 37] or [3, p. 40]), which contradicts the fact that B_1 is a connected set.

From $\Re \subset G(t, \Re) \subset G(t, B_1)$, we have $\{G(t, B_1)\} \cap A_1 \neq \emptyset$, $\{G(t, B_1)\} \cap A_2 \neq \emptyset$. But $A_1 \cup A_2$ does not cover $G(t, B_1)$ for any $t \geq 0$. Thus, there exist $\xi_n \in G(t_n, B_1)$, where $t_n \to +\infty$, such that $\xi_n \notin A_1 \cup A_2$. As G is asymptotically upper semicompact, we obtain that the sequence $\{\xi_n\}$ has a converging subsequence and its limit ξ belongs to $\omega(B_1) \subset \Re$ but does not belong to $A_1 \cup A_2$, which is a contradiction. □

Remark 1.9. In Theorem 1.6, it is crucial to assume that the map $G\,(t, \cdot)$ is upper semicontinuous.

Let $D \subset X$ be a Hausdorff topological space. We shall denote the closure of $A \subset D$ in D by $cl_D A$. The set $K \subset D$ is called (X, D)-attracting if $\forall B \in \mathscr{B}(X)$, $\exists T(B) \in \Gamma_+$ such that $G(t, B) \subset D$, $\forall t \geq T(B)$, and for any neighborhood $O(K)$ of K (in D), there exists $T \in \Gamma_+$ such that $G(t, B) \subset O(K)$, $\forall t \geq T$.

The set $K \subset D$ is called X-absorbing if $\forall B \in \mathscr{B}(X)$, $\exists T(B) \in \Gamma_+$ such that $G(t, B) \subset K$, $\forall t \geq T(B)$.

Definition 1.9. The set $\Re$ is called a (X, D)-attractor for the m-semiflow G if it is (X, D)-attracting and negatively semiinvariant.

Theorem 1.7. *Let for any $t \in \Gamma_+ \backslash \{0\}$, $G(t, \cdot) : X \to P(D)$, and let there exist an X-absorbing set $K \subset D$. Suppose also that K is compact in D and bounded in X. Assume, moreover, that $\forall S \subset K$*

$$cl_D G(t, S) \subset G(t, cl_D S), \ \forall t \in \Gamma_+$$

and also that the set $\{y \in X \mid G(t, y) \cap \{y_0'\} \neq \emptyset\} \cap K$ is compact in D, $\forall y_0 \in D$.

Then the m-semiflow G has the global (X, D)-attractor $\Re$, which is compact in D and bounded in X. It is the minimal closed set (in D) which attracts any $B \in \mathscr{B}(X)$.

Proof. For any $B \in \mathscr{B}(X)$, $\exists\, T(B) \in \Gamma_+$ such that $G(t, B) \subset K$, $\forall t \geq T(B)$. We set $\Re = \omega(K) = \bigcap_{\tau \geq t_0} cl_D \gamma_\tau^+(K)$, where $t_0 = T(K)$. We shall prove that $\Re$ is a global (X, D)-attractor. It is clear that $\Re \in \mathscr{B}(X)$ if it is nonempty. Let us prove that $\Re$ is nonempty, compact in D and (X, D)-attracting. We note that $cl_D \gamma_\tau^+(K)$ is compact in D for any $\tau \geq t_0$ and then $\Re$ is nonempty and compact in D. Since $G(t + T(B), B) \subset G(t, G(T(B), B) \subset G(t, K)$, in order to prove the attracting property, it is sufficient to obtain that $\Re$ attracts K. Suppose that $\Re$ does not attract K, that is, there exists an open neighborhood $O(\Re)$ such that $\forall T$, $\exists t \geq T$ for which $G(t, K) \cap (D \backslash O(\Re)) \neq \emptyset$. Then there exist nets $t_\sigma \to \infty$, $\xi_\sigma \in G(t_\sigma, K) \cap (D \backslash O(\Re)) = G(t_\sigma, K) \cap (K \backslash O(\Re))$ such that $\xi_\sigma \to \xi$ in D. Since $\xi \in cl_D \gamma_\tau^+(K)$, $\forall \tau > 0$, we have $\xi \in \Re$. On the other hand, $\xi \in cl_D(K \backslash O(\Re)) = K \backslash O(\Re)$, which is a contradiction. It remains to prove that $\Re$ is negatively semiinvariant. Let us prove first the next equality

$$\bigcap_{\tau \geq t_0} G(t, cl_D \gamma_\tau^+(K)) = G(t, \bigcap_{\tau \geq t_0} cl_D \gamma_\tau^+(K)) \tag{1.3}$$

Let $\xi \in \bigcap_{\tau \geq t_0} G(t, cl_D \gamma_\tau^+(K))$. Then $\forall \tau \geq t_0$, $\exists x_\tau \in cl_D \gamma_\tau^+(K)$ such that $\xi \in G(t, x_\tau)$. Let $\tau_\sigma \to \infty$. We can assume that $x_{\tau_\sigma} \to x$ in D, $x \in \mathfrak{R}$. Hence, $x_{\tau_\sigma} \in \{y \in X \mid G(t, y) \ni \xi\}$ and, on the other hand, $x_{\tau_\sigma} \in cl_D \gamma_\tau^+(K) \subset K$, that is, $x_{\tau_\sigma} \in \{y \in X \mid G(t, y) \ni \xi\} \cap K$. In view of the compactness of this set $x \in \{y \in X \mid G(t, y) \ni \xi\}$, that is, $\xi \in G(t, x) \subset G(t, \mathfrak{R}) = G(t, \bigcap_{\tau \geq t_0} cl_D \gamma_\tau^+(K))$. Hence, $\bigcap_{\tau \geq t_0} G(t, cl_D \gamma_\tau^+(K)) \subset G(t, \bigcap_{\tau \geq t_0} cl_D \gamma_\tau^+(K))$. Since the inverse inclusion is obvious, (1.3) holds. Hence, we have

$$\begin{aligned} G(t, \mathfrak{R}) &= G(t, \bigcap_{\tau \geq t_0} cl_D \gamma_\tau^+(K)) = \bigcap_{\tau \geq t_0} G(t, cl_D \gamma_\tau^+(K)) \\ &\supset \bigcap_{\tau \geq t_0} cl_D G(t, \gamma_\tau^+(K)) = \bigcap_{\tau \geq t_0} cl_D \bigcup_{s \geq \tau} G(t, G(s, K)) \\ &\supset \bigcap_{\tau \geq t_0} cl_D \bigcup_{s \geq \tau} G(t + s, K) = \bigcap_{\tau \geq t + t_0} cl_D \gamma_\tau^+(K) = \mathfrak{R}. \end{aligned}$$

Let $Z \subset D$ be (X, D)-attracting. Then for any neighborhood $O(Z)$, there exists T such that $\mathfrak{R} \subset G(t, \mathfrak{R}) \subset O(Z)$, $\forall t \geq T$. Hence, $\mathfrak{R} \subset cl_D Z$. Therefore, $\mathfrak{R}$ is the minimal closed set attracting each $B \in \mathscr{B}(X)$. □

Remark 1.10. If $D = X$, then $\mathfrak{R}$ is a global compact attractor of the m-semiflow G. This result was proved in [4].

Remark 1.11. If in Theorem 1.7, $G(t_1 + t_2, x) = G(t_1, G(t_2, x))$, $\forall t_1, t_2 \in \Gamma_+$, $\forall x \in X$, then the global attractor $\mathfrak{R}$ is invariant.

Proof. We know that $\mathfrak{R} \subset G(t, \mathfrak{R})$, $\forall t \in \Gamma_+$. Let us prove the inverse. For any $t, \tau \in \Gamma_+$, we have

$$G(t, \mathfrak{R}) \subset G(t, G(\tau, \mathfrak{R})) \subset G(t + \tau, \mathfrak{R})$$

For any neighborhood $O(\mathfrak{R})$ of $\mathfrak{R}$ in D, there exists T such that $G(t + \tau, \mathfrak{R}) \subset O(\mathfrak{R})$, $\forall \tau \geq T$. Since $O(\mathfrak{R})$ is arbitrary, we have $G(t, \mathfrak{R}) \subset cl_D \mathfrak{R} = \mathfrak{R}$. Hence, $\mathfrak{R} = G(t, \mathfrak{R})$, $\forall t \in \Gamma_+$. □

Proposition 1.3. *Suppose that the conditions of Theorem 1.7 hold. Assume also that the set K is connected in D and that $G(t, \cdot) : K \to P(D)$ is upper semicontinuous and has connected values for each $t > 0$ (with respect to the topology of D). Then the global attractor $\mathfrak{R}$ is connected in D.*

Proof. Let $\mathfrak{R}$ be not connected. Then $\mathfrak{R} = A_1 \cup A_2$, where A_i are compact sets in D, and there exist neighborhoods $O(A_1)$, $O(A_2)$ such that $O(A_1) \cap O(A_2) = \emptyset$. The set $O(\mathfrak{R}) = O(A_1) \cup O(A_2)$ is a neighborhood of $\mathfrak{R}$. There exists T such that $G(t, K) \subset O(\mathfrak{R})$, $\forall t \geq T$. It is well known (see the proof of Theorem 1.6 or [15, 33]) that an upper semicontinuous map with connected values maps any connected set into a connected one. Hence, $G(t, K)$ belongs completely to one of the sets $O(A_i)$. Therefore, $\Re \subset G(t, \Re) \subset G(t, K) \subset O(A_i)$, which is a contradiction. □

Lemma 1.6. *Let $G(t, \cdot) : X \to C(X)$ be an upper semicontinuous m-semiflow and let $D \subset X$ with continuous injection. If the set K is compact in D and $G(t, K) \subset D$, $\forall t \in \Gamma_+$ then the last two conditions of Theorem 1.7 hold, that is, $cl_D G(t, M) \subset G(t, cl_D M)$, $\forall M \subset K$, and $\{y \in X \mid G(t, y) \ni y_0\} \cap K$ are compact in D, $\forall y_0 \in D$.*

Proof. Let $\xi \in cl_D G(t, M)$. Then there exist nets $x_\sigma \in M$, $\xi_\sigma \in G(t, x_\sigma)$ such that $\xi_\sigma \to \xi$ in D. Since $cl_D M$ is compact in D, there is no loss of generality in assuming $x_\sigma \to x$ in D and $x \in cl_D M$. Since $G(t, \cdot) : X \to C(X)$ is upper semicontinuous, $G(t, \cdot)$ is closed in X (see [3, p. 42]). Hence, as the injection $D \subset X$ is continuous, we have $\xi \in G(t, cl_D M)$. Let us consider now an arbitrary element $y_0 \in D$. The inverse image of y_0, $G^{-1}(y_0) = \{y \in X \mid G(t, y) \ni y_0\}$ is closed in X, because $G(t, \cdot)$ is upper semicontinuous (see [3, p. 40]). Hence, $G^{-1}(y_0)$ is closed in D and then $G^{-1}(t, y_0) \cap K$ is compact in D. □

Remark 1.12. Theorems 1.1, 1.3, 1.4, and 1.7 generalize to the multivalued case similar results in the single-valued case (see [7, 40, 53, 54]).

Remark 1.13. In [77] were proved some theorems concerning existence of attractors for multivalued semigroups. However, the author does not assume the semigroup to satisfy the property $G(t_1 + t_2, x) \subset G(t_1, G(t_2))$. Hence, it is not possible in this case to obtain that the global attractor is negatively semiinvariant.

1.2 Comparison Between Trajectory and Global Attractors for Evolution Systems

We have seen in the introduction to this chapter that there are three methods for studying the asymptotic behavior of solutions of differential equations without uniqueness of solutions: the method of multivalued semiflows, the method of generalized semiflows, and the theory of trajectory attractors.

It is quite natural and interesting to make a comparison of the method of trajectory attractors with the method of multivalued semiflows. In the case of uniqueness of solutions, some results about the relationship between trajectory and global attractors for semigroups can be found in [25, 26].

The aim of this section is to give a complete comparison between the approach of multivalued semiflows and the approach of trajectory attractors in the useful case, that is, when uniqueness of the Cauchy problem does not hold. *Also, we apply the general results to reaction-diffusion equations and the nonlinear wave equation.* These results are borrowed from [50].

1.2.1 The Main Definitions

Let E, E_0 be Banach spaces, $E \subseteq E_0$ with continuous embedding (it is possible that $E = E_0$), and let

$$W = L_\infty(\mathbf{R}; E) \cap \mathbf{C}^{\mathrm{loc}}(\mathbf{R}; E_0),$$
$$W^+ = L_\infty(\mathbf{R}_+; E) \cap \mathbf{C}^{\mathrm{loc}}(\mathbf{R}_+; E_0)$$

We note that if $f(\cdot) \in W^+$, then $f(\cdot) \in \mathbf{C}_w^{\mathrm{loc}}(\mathbf{R}_+; E)$ and

$$\|f(t)\|_E \le \|f\|_{L_\infty(\mathbf{R}_+;E)}, \forall t \ge 0.$$

Let K^+ be some set of solutions of an evolution system (usually we shall work with evolution equations in partial derivatives) satisfying the following properties:

$(K1)$ $K^+ \subset W^+$.
$(K2)$ For any $z \in E$ there exists $\varphi(\cdot) \in K^+$ such that $\varphi(0) = z$.
$(K3)$ $\varphi_\tau(\cdot) := \varphi(\cdot + \tau) \in K^+, \forall\ \tau \ge 0$.

It is important to notice that we do not assume any concatenation or continuity properties for the elements from K^+ as in the definition of generalized semiflows [9]. Let us consider the translation semigroup $\{T(t)\}_{t\ge 0}$ given by

$$T(t)\varphi(\cdot) = \varphi(\cdot + t), \forall\ \varphi(\cdot) \in K^+.$$

From $(K3)$, we obtain

$$T(t)K^+ \subset K^+, \quad \forall\ t \ge 0.$$

We note that

$$f_n(\cdot) \to f(\cdot) \text{ in } \mathbf{C}^{\mathrm{loc}}(\mathbf{R}_+; E_0) \Longleftrightarrow$$
$$f_n(\cdot) \to f(\cdot) \text{ in } \mathbf{C}([0, M]; E_0), \forall M > 0.$$

If Π_M is the operator of restriction on $[0, M]$, and Π_+ is the operator of restriction on $[0, +\infty)$, then

$$Y \text{ is compact in } \mathbf{C}^{\mathrm{loc}}(\mathbf{R}_+; E_0) \Longleftrightarrow$$
$$\Pi_M Y \text{ is compact in } \mathbf{C}([0, M]; E_0), \forall M > 0.$$

As before, for an arbitrary metric space Y, we denote by $dist_Y(A, B)$ the Hausdorff semidistance from the set A to the set B, defined by

$$dist_Y(A, B) = \sup_{a\in A} \inf_{b\in B} \rho(a, b),$$

where $\rho(\cdot, \cdot)$ is the metric in Y.

We will use now the notation and definitions introduced in [26].

Definition 1.10. The set $U \subset K^{+}$ is called a *trajectory attractor* (with respect to the space of trajectories K^{+} in the topology $\mathbf{C}^{\mathrm{loc}}(\mathbf{R}_{+}; E_0)$), if:

1. U is compact in $\mathbf{C}^{\mathrm{loc}}(\mathbf{R}_{+}; E_0)$ and bounded in $L_{\infty}(\mathbf{R}_{+}; E)$.
2. U is invariant, that is, $T(t)U = U$, $\forall t \geq 0$.
3. U is an attracting set, that is, for every set $B \subset K^{+}$, bounded in $L_{\infty}(\mathbf{R}_{+}; E)$, we have that for any $M > 0$,

$$dist_{\mathbf{C}([0,M];E_0)}(\Pi_M\, T(t)\, B, \Pi_M\, U) \to 0, \tag{1.4}$$

as $t \to \infty$.

Definition 1.11. The function $\varphi(\cdot) \in W$ is called a *complete trajectory* for K^{+}, if

$$\Pi_{+}\varphi_h(\cdot) \in K^{+}, \forall\, h \in \mathbf{R},$$

where $\varphi_h(s) = \varphi(s+h)$.

Let $\mathbf{K}$ be the union of all complete trajectories for K^{+}.

Theorem 1.8. *[26] If there exists an attracting set $P \subset K^{+}$, which is compact in $\mathbf{C}^{\mathrm{loc}}(\mathbf{R}_{+}; E_0)$ and bounded in $L_{\infty}(\mathbf{R}_{+}; E)$, then there exists a trajectory attractor $U \subseteq P$ and*

$$U = \Pi_{+}\mathbf{K}. \tag{1.5}$$

As before, let us denote by $P(E)$ ($\beta(E)$, $C(E)$) the set of all nonempty (nonempty bounded, nonempty closed) subsets of E. The following theorem is proved in a similar way as in the previous section, where the particular case $E = E_0$ is considered (see also [47, 51]). We consider a multivalued semiflow $G : \mathbf{R}_{+} \times E \to P(E)$ as given in Definition 1.1 of the previous section.

Definition 1.12. The set $A \subset E$ is called a global (E, E_0)-*attractor* for G, if:

1. A is compact in E_0 and bounded in E;
2. $A \subseteq G(t, A)$, $\forall\, t \geq 0$.
3. For every bounded $B \subset E$

$$dist_{E_0}(G(t, B), A) \to 0,\ t \to \infty.$$

Of course, the global (E, E_0)-attractor is unique and the minimal closed set (in E_0) attracting every bounded set of E. We put

$$\omega(B) = \bigcap_{s \geq 0} cl_{E_0}(\bigcup_{t \geq s} G(t, B)).$$

It is wellknown (see Lemma 1.1) that $y \in \omega(B)$ if and only if there exist $t_n \nearrow \infty$ and $\xi_n \in G(t_n, B)$ such that

$$\xi_n \to y \text{ in } E_0.$$

Theorem 1.9. *Let the m-semiflow G have an attracting set B_0, which is bounded in E and compact in E_0. That is, for every $B \in \beta(E)$, we have*

$$dist_{E_0}(G(t, B), B_0) \to 0,\ t \to \infty.$$

Also, let for any $t \geq 0$ the map $E_0 \ni z \mapsto G(t,z) \subset E_0$ have closed graph.

Then G has the global (E, E_0)-attractor

$$A = \bigcup_{B \in \beta(E)} \omega(B) = \omega(B_0).$$

If, moreover, the m-semiflow G is strict, then A is invariant, that is, $A = G(t, A)$, for all $t \geq 0$.

On the other hand, if A is a global (E, E_0)-attractor for G, then G is asymptotically compact, that is, for every $t_n \nearrow \infty$, $B \in \beta(E)$, an arbitrary sequence $\xi_n \in G(t_n, B)$ is precompact in E_0. Also, if, additionally, for any $t \geq 0$ the map $z \mapsto G(t,z)$ has closed graph in $A \subset E_0$ (in the sense that if $z_n \to z \in A$, $y_n \to y \in A$ in E_0, where $y_n \in G(t, z_n)$, imply $y \in G(t, z)$), then $A = \cup_{B \in \beta(E)} \omega(B) = \omega(A)$.

Proof. First, for any $B \in \beta(E)$, the set $\omega(B)$ is nonempty negatively semiinvariant, compact in E_0 and bounded in E. Also, it is the minimal closed set (in E_0) attracting B.

Take a sequence $\xi_n \in G(t, B)$. Then

$$dist_{E_0}(\xi_n, B_0) \to 0$$

and the compactness of B_0 implies that ξ_n is a precompact sequence in E_0. Hence, $\omega(B)$ is nonempty. Also, as it is obvious that $\omega(B) \subset B_0$ and $\omega(B)$ is a closed set in E_0, we obtain that $\omega(B)$ is compact in E_0 and bounded in E.

The facts that $\omega(B)$ is negatively semiinvariant and the minimal closed set (in E_0) attracting B can be proved in the same way as in Theorem 1.1.

It is clear then that $A = \bigcup_{B \in \beta(E)} \omega(B) \subset B_0$ attracts every bounded set of E and is bounded in E. Also, it follows easily that $A \subset G(t, A)$ and that it is minimal. Let us prove that $A \subset \omega(B_0)$. Indeed, since

$$\omega(B) \subset G(t, \omega(B)) \subset G(t, B_0),$$

we have

$$dist_{E_0}(\omega(B), \omega(B_0))$$
$$\leq dist_{E_0}(G(t, B_0), \omega(B_0)) \to 0$$

as $t \to \infty$, and the result follows. Therefore, $A = \omega(B_0)$, so that A is compact in E_0.

The invariance of A if the semiflow is strict is proved exactly in the same way as in Remark 1.7.

Finally, arguing as in the first part of the theorem it is easy to show that for any $B \in \beta(E)$ the set $\omega(B)$ is non-empty, compact in E_0 and the minimal closed set (in E_0) attracting B. Also, since $\omega(B) \subset A$, the fact that the graph of $G(t,\cdot)$ is closed in $A \subset E_0$ implies that $\omega(B)$ is negatively semi-invariant. It follows then that $\overline{A} = \cup_{B\in\beta(X)}\omega(B) = \omega(A)$ is also a (E, E_0)-global attractor, and as the global (E, E_0)-attractor is unique, we have $A = \overline{A}$.

1.2.2 Main Results

Now, let K^+ be some set of solutions of an evolution system satisfying properties (K1)–(K3) given in the previous section. Then we can correctly construct the following map $G : \mathbf{R}_+ \times E \mapsto P(E)$:

$$G(t, z) = \{\varphi(t) \ : \ \varphi(\cdot) \in K^+, \ \ \varphi(0) = z\}. \tag{1.6}$$

Lemma 1.7. *The map G, defined by (1.6), is an m-semiflow.*

Proof. According to (1.6), for every $\xi \in G(t + s, z)$ we have $\xi = \varphi(t + s)$, where $\varphi(\cdot) \in K^+, \ \ \varphi(0) = z$. So $\varphi(s) \in G(s, z)$. Let us consider $v(\cdot) = T(s)\varphi(\cdot) = \varphi(\cdot + s)$. From property (K3), we obtain $v(\cdot) \in K^+$ and $v(0) = \varphi(s)$. So $v(t) = \varphi(t + s) = \xi \in G(t, v(0)) = G(t, \varphi(s) \subset G(t, G(s, z))$. On the other hand, it is obvious that $G(0, z) = z$, $\forall\, z \in E$. □

Remark 1.14. By the way we have proved that $\varphi(t + s) \in G(t, \varphi(s))$, $\forall\ \varphi(\cdot) \in K^+, \ \ \forall\, t, s \geq 0$

Remark 1.15. It does not follow that the m-semiflow G is strict, as the concatenation function

$$\Theta(\tau) = \begin{cases} v(\tau), & 0 \leq \tau \leq s, \\ u(\tau - s), & \tau > s, \end{cases}$$

where $v(\cdot), u(\cdot) \in K^+$, $v(s) = u(0)$ can be in general a function not belonging to K^+.

The following result generalizes Corollary 2.1 in [26] to the multivalued case.

Theorem 1.10. *Let $U \subset K^+$ be a trajectory attractor in the space of trajectories K^+ in the topology $\mathbf{C}^{\mathrm{loc}}(\mathbf{R}_+; E_0)$ and the following condition holds:*

$$\begin{array}{c} \textit{for every } B \in \beta(E), \textit{ the set} \\ \widetilde{B} = \{\varphi(\cdot) \in K^+ \mid \varphi(0) \in B\} \\ \textit{is bounded in } L_\infty(\mathbf{R}_+; E). \end{array} \tag{1.7}$$

Then the m-semiflow G, defined by (2), has the global (E, E_0)-attractor A and

$$A = U(0). \tag{1.8}$$

Proof. It is enough to prove that the set A, defined by (1.8), satisfies the conditions of Definition 1.12.

U is bounded in $L_\infty(\mathbf{R}_+; E)$, which implies the existence of $C > 0$ such that $\|u\|_{L_\infty(\mathbf{R}_+;E)} \le C$, $\forall\, u(\cdot) \in U$, and then, $\|u(t)\|_E \le C$, $\forall\, t \ge 0$. Hence, $\|u(0)\|_E \le C$, so that A is bounded in E.

As U is compact in $\mathbf{C}^{\text{loc}}(\mathbf{R}_+; E_0)$, so $A = U(0)$ is compact in E_0.

Let $\xi \in A$ be arbitrary, so that $\xi = u(0)$, where $u(\cdot) \in U = T(t)U$. Hence, there exists $\varphi(\cdot) \in U$ such that $u(\cdot) = T(t)\varphi(\cdot) = \varphi(t + \cdot)$, and then,

$$\xi = u(0) = \varphi(t) \in G(t, \varphi(0)) \subset G(t, A).$$

It follows that

$$A \subset G(t, A), \ \forall\, t \ge 0.$$

Let $B \in \beta(E)$ be arbitrary. Then the corresponding set $\widetilde{B} \subset K^+$ from (1.7) is bounded in $L_\infty(\mathbf{R}_+; E)$, so

$$dist_{\mathbf{C}([0,1];E_0)}(\Pi_1 T(t)\widetilde{B}, \Pi_1 U) \to 0, \ \ t \to \infty.$$

Thus,

$$dist_{E_0}(T(t)\widetilde{B}|_{s=0}, U(0)) \to 0, \ \ t \to \infty,$$

and, finally,

$$dist_{E_0}(G(t, B), A) \to 0, \ \ t \to \infty.$$

It means that G has the global (E, E_0)-attractor $A = U(0)$. □

Remark 1.16. In Theorem 2.1 from [26], the formula $A = U(0)$ was established in the multivalued case for a different type of global attractor, which is defined as the minimal set attracting the sections $B(t)$ of every set of trajectories $B \subset K^+$, bounded in $L_\infty(\mathbf{R}_+; E)$.

Corollary 1.2. *Since U is invariant, we have $A = U(0) = U(t)$, $\forall\, t \ge 0$.*

Theorem 1.11. *If U is a trajectory attractor and A is a global (E, E_0)-attractor, then $U(0) \subseteq A$.*

If, moreover, A is invariant, that is $A = G(t, A)$, $\forall\, t \ge 0$ (in particular, this holds if G is a strict m-semiflow), then $U(0) = A$.

Proof. Arguing as in the proof of Theorem 1.10, we can deduce that the set $\tilde{A} = U(0)$ is bounded in E, compact in E_0, and $\tilde{A} \subset G(t, \tilde{A})$ for all $t \ge 0$. From the definition of global (E, E_0)-attractor, we have

$$dist_{E_0}(G(t, \tilde{A}), A) \to 0, \ as\, t \to \infty.$$

So $\tilde{A} = U(0) \subseteq A$.

Let A be invariant and let us consider the set

$$\widetilde{B} = \{\varphi(\cdot) \in K^+ \mid \varphi(0) \in A\}.$$

Then $\varphi(t) \in G(t, A) = A$, $\forall\, t \geq 0$, $\forall\, \varphi(\cdot) \in \widetilde{B}$. So, the set $\widetilde{B}$ is bounded in $L_\infty(\mathbf{R}_+; E)$, and arguing as in the proof of Theorem 1.10, we get

$$dist_{E_0}(T(t)\widetilde{B}|_{s=0}, \tilde{A}) \to 0, \;\; t \to \infty.$$

Therefore,

$$dist_{E_0}(G(t, A), \tilde{A}) \to 0, \;\; t \to \infty.$$

Then $A \subseteq \tilde{A}$ and, finally, $A = \tilde{A} = U(0)$. □

Now, we give an example which shows the importance of condition (1.7).

Example 1.1. Let $E = E_0 = \mathbf{R}_+$ and let us consider the following differential inclusion

$$\dot{u}(t) \leq u^2(t) + 2, \tag{1.9}$$

or equivalently

$$\dot{u}(t) \in (-\infty, u^2(t) + 2]. \tag{1.10}$$

The function $u(\cdot)$ is called a solution of (1.10) if it an absolutely continuous function and satisfies (1.9) almost everywhere.

We define

$$\begin{aligned} K^+ = \{u(\cdot) \text{ is a solution of (1.10)},\; u(\cdot) \in L_\infty(\mathbf{R}_+),\\ u(t) \geq 0, \text{and for any } \tau \geq 0,\; \text{we have}\\ u(t+\tau) = 0,\; \forall\, t \geq \sup_{s \geq \tau} u(s)\}. \end{aligned}$$

Then we immediately obtain that for any $u(\cdot) \in K^+$,

$$u(t) = 0, \;\; \forall\, t \geq \|u\|_\infty := \sup_{t \geq 0} |u(t)|\,.$$

Moreover, $K^+ \subset W^+$ and $K^+ \neq \oslash$, because $0 \in K^+$. Let us verify conditions (K2), (K3).

For any $u_0 \in \mathbf{R}_+$, there exists $u(\cdot) \in K^+$ such that $u(0) = u_0$. For example,

$$u(t) = \begin{cases} -t + u_0, & t \in [0, u_0], \\ 0, & t \geq u_0. \end{cases}$$

Indeed, $u(\cdot)$ is a solution, $u(\cdot) \in L_\infty(\mathbf{R}_+)$, $\|u\|_\infty = u_0$, and for any $\tau \geq 0$, we get $u(t+\tau) = 0$, $\forall\, t \geq \sup_{s \geq \tau} u(s) = u(\tau)$.

Finally, for all $\tau_0 \geq 0$ and all $u(\cdot) \in K^+$, we have $v(\cdot) = u(\cdot + \tau_0) \in K^+$, as $v(\cdot)$ is a solution, $v(\cdot) \in L_\infty(\mathbf{R}_+)$, $v(t) \geq 0$ and for any $\tau \geq 0$, we have $v(t+\tau) = u(t+\tau+\tau_0) = 0$, $\forall\, t \geq \sup_{s\geq\tau} u(s+\tau_0) = \sup_{s\geq\tau} v(s)$.

So, K^+ satisfies conditions (K1)–(K3). Further, for any set $B \subset K^+$, bounded in $L_\infty(\mathbf{R}_+)$, we can find a constant $c > 0$ such that $\|u\|_\infty \leq c$, $\forall\, u(\cdot) \in B$. So, $u(t) = 0$, $\forall\, t \geq c$, and then for any $M > 0$, we have

$$\max_{s\in[0,M]} |T(t)u(s) - 0| = \max_{s\in[0,M]} |u(s+t)| = 0,$$

for all $t \geq c$. Therefore, $U = \{0\}$ is a trajectory attractor in the space of trajectories K^+.

On the other hand, the set

$$\{u(\cdot) \in K^+ \ : u(0) = u_0\}$$

is unbounded in $L_\infty(\mathbf{R}_+)$ for every $u_0 > 0$ (in particular, condition (1.7) does not hold!). Indeed, if for every $N > u_0$ we consider the following absolutely continuous function

$$u_N(t) = \begin{cases} \frac{1}{\frac{1}{u_0} - t}, & t \in [0, \frac{1}{u_0} - \frac{1}{N}], \\ \frac{N}{N - \frac{1}{u_0} + \frac{1}{N}}(N-t), & t \in \left[\frac{1}{u_0} - \frac{1}{N}, N\right], \\ 0, & t \geq N, \end{cases}$$

then we have that $u_N(0) = u_0$, $u_N(\cdot)$ is a solution, $u_N(\cdot) \in L_\infty(\mathbf{R}_+)$, $u_N(t) \geq 0$, $\|u_N\|_\infty = N$, and

$$\max_{s\geq\tau} u_N(s) = \begin{cases} N, \ \tau \in [0, \frac{1}{u_0} - \frac{1}{N}], \\ u_N(\tau) \geq N - \tau, \ \tau \in \left[\frac{1}{u_0} - \frac{1}{N}, N\right], \\ 0, \quad \tau \geq N. \end{cases}$$

So, for any $\tau \geq 0$, we have $u_N(t+\tau) = 0$, $\forall\, t \geq \sup_{s\geq\tau} u_N(s)$, and, therefore,

$$u_N(\cdot) \in \{u(\cdot) \in K^+ \ : u(0) = u_0\},$$

but $\|u_N\|_\infty \to \infty$, as $N \to \infty$.

If we take $t_N = \frac{N}{2} \nearrow \infty$, then for $\xi_N = u_N(t_N) \in G(t_N, u_0)$, we have $\xi_N \to \infty$, and from the second part of Theorem 1.9, we deduce that G has no global attractor.

Theorem 1.12. *Let A be a global (E, E_0)-attractor of the m-semiflow G, defined by (1.6), and let the following condition holds:*

$$\begin{array}{c} \textit{for any } \{\varphi_n(\cdot)\} \subset K^+ \textit{satisfying} \\ \varphi_n(0) \to \varphi_0 \in A \textit{ in } E_0, \\ \textit{there exists } \varphi(\cdot) \in K^+,\ \varphi(0) = \varphi_0, \\ \textit{such that for some subsequence} \\ \varphi_n(t) \to \varphi(t) \textit{ in } E_0,\ \forall\, t \geq 0. \end{array} \tag{1.11}$$

Then the formula

$$U = \Pi_+\{\varphi(\cdot) \in \mathbf{K} \,:\, \varphi(0) \in A\} = \Pi_+\mathbf{K} \tag{1.12}$$

defines a trajectory attractor in the space of trajectories K^+.

Proof. We shall split the proof in several lemmas.

Lemma 1.8. *The second equality in (1.12) is true. Moreover, for any $\varphi \in \mathbf{K}$, we have that $\varphi(t) \in A$, for any $t \in \mathbf{R}$.*

Proof. Note that for every $\varphi(\cdot) \in \mathbf{K}$, we have $\varphi(\cdot) \in W$, so that there exists $c > 0$ such that $\|\varphi(s)\|_E \leq c$, $\forall\, s \in \mathbf{R}$. Then $B = \bigcup_{s\in\mathbf{R}} \varphi(s)$ is bounded in E. Moreover, for any $t \geq 0$, $s \in \mathbf{R}$, $\varphi(\cdot) \in \mathbf{K}$,

$$\begin{aligned} \varphi(s+t) = \varphi_s(t) &\in G(t, \varphi_s(0)) \\ &= G(t, \varphi(s)) \subset G(t, B). \end{aligned}$$

Then as A attracts B, for any $\tau \in \mathbf{R}$ and any $\varepsilon > 0$, one can choose s and t such that

$$dist_{E_0}(\varphi(\tau), A) = dist_{E_0}(\varphi(s+t), A) < \varepsilon.$$

So, $\varphi(\tau) \in A$, $\forall\, \tau \in \mathbf{R}$. In particular, $\varphi(0) \in A$. Therefore, the second equality in (1.6) is true. □

Now, we prove that the set U is nonempty and bounded.

Lemma 1.9. *The set U, defined by (1.12), is nonempty and bounded in $L_\infty(\mathbf{R}_+; E)$.*

Proof. Let $z \in A$ be arbitrary. We note that condition (1.11) implies that the graph of $G\,(t, \cdot)$ is closed in $A \subset E_0$. Then from the second part of Theorem 1.9, $z \in \omega(A)$. So there exist $\xi_n \in G(t_n, A)$ such that $\xi_n \to z$ in E_0. Therefore, there exist $\varphi_n(\cdot) \in K^+$, for which $\varphi_n(t_n) = \xi_n$ and $\varphi_n(0) \in A$.

Let us consider $\psi_n^0(\cdot) = \varphi_n(\cdot + t_n) \in K^+$. Then $\psi_n^0(0) \to z$ in E_0, so that from (1.11), there exists $\psi^0(\cdot) \in K^+$ satisfying $\psi^0(0) = z$ such that up to a subsequence $\psi_n^0(t) \to \psi^0(t)$, $\forall t \geq 0$. Since $\psi_n^0(t) = \varphi_n(t + t_n)$, by Theorem 1.9 $\psi^0(t) \in A$, for all $t \geq 0$.

Further, we consider $\psi_n^1(\cdot) = \varphi_n(\cdot + t_n - t_1) \in K^+$. As $t_n - t_1 \to \infty$, the sequence $\psi_n^1(0) = \varphi_n(t_n - t_1)$ has a subsequence converging to some element of A. Thus, by (1.11), there exists $\psi^1(\cdot) \in K^+$ such that for some subsequence $\psi_n^1(t) \to \psi^1(t)$, $\forall\, t \geq 0$, and $\psi^1(t) \in A$, $\forall\, t \geq 0$. Also, from the equality

$$\psi_n^1(t+t_1)=\psi_n^0(t),\ \forall\, t\geq 0,$$

we have

$$\psi^1(t+t_1)=\psi^0(t),\ \forall\, t\geq 0.$$

Arguing inductively we obtain that for any $k\geq 1$, there exists a function $\psi^k(\cdot)\in K^+$ such that $\psi^k(t)\in A$, for any $t\geq 0$, and

$$\psi^k(t+t_k-t_{k-1})=\psi^{k-1}(t),\ \forall\, t\geq 0.$$

For any $t\in \mathbf{R}$, we put

$$\psi(t):=\psi^k(t+t_k), \text{if } t\geq -t_k.$$

Then $\psi(\cdot)$ is correctly defined, as for $t\geq -t_k$,

$$\psi^{k+1}(t+t_{k+1})=\psi^{k+1}(t+t_k-t_k+t_{k+1})=\psi^k(t+t_k).$$

So $\psi(\cdot)\in \mathbf{C}^{\mathrm{loc}}(\mathbf{R};E_0)$ and from $\psi^k(t)\in A$, $\forall\, t\geq 0$, we have that $\psi(t)\in A$, $\forall\, t\in\mathbf{R}$. In particular, $\psi(\cdot)\in W$.

Moreover, for any $h\in\mathbf{R}$, there exists t_k such that $h+t_k\geq 0$. Hence, for any $s\geq 0$, we have $s+h\geq -t_k$ and $\psi_h(s)=\psi(s+h)=\psi^k(s+h+t_k)$. Thus, $\Pi_+\psi_h(\cdot)=T(h+t_k)\psi^k(\cdot)\in K^+$ and $\psi(\cdot)$ are a complete trajectory such that $\psi(0)=\psi^k(t_k)=\psi^{k-1}(t_{k-1})=\ldots=\psi^0(0)=z$. So $\mathbf{K}\neq\emptyset$ and $U\neq\emptyset$.

Finally, by Lemma 1.8, for all $\varphi(\cdot)\in\mathbf{K}$ and $\tau\in\mathbf{R}$, we have $\varphi(\tau)\in A$, so that U is bounded in $L_\infty(\mathbf{R}_+;E)$. □

Let us prove now the compacity of U in $\mathbf{C}^{\mathrm{loc}}(\mathbf{R}_+;E_0)$.

Lemma 1.10. *The set U is compact in $\mathbf{C}^{\mathrm{loc}}(\mathbf{R}_+;E_0)$.*

Proof. Let $\{\varphi_n(\cdot)\}\subset U$ be arbitrary. Then $\{\varphi_n(0)\}\in A$ is precompact in E_0, so that passing to a subsequence $\varphi_n(0)\to\varphi_0\in A$ in E_0. Thus, from (1.11), there exists $\varphi(\cdot)\in K^+$ with $\varphi(0)=\varphi_0$ such that $\varphi_n(t)\to\varphi(t)$, for any $t\geq 0$. Since $\varphi_n(\cdot),\varphi(\cdot)\in\mathbf{C}^{\mathrm{loc}}(\mathbf{R}_+;E_0)$, according to Theorem 2.2 in [9], we can deduce that $\varphi_n(\cdot)\to\varphi(\cdot)$ in $\mathbf{C}([\delta,M];E_0)$, $\forall\,\delta>0$, $M>\delta$. (In fact, the trajectories in Theorem 2.2 from [9] are assumed to satisfy also the concatenation property, but in the proof, this fact is not used.).

Now we shall use again some arguments borrowed from [9] in order to prove that for $t_n\to 0$, we have $\varphi_n(t_n)\to\varphi_0$. If it is not true, then there exist $\varepsilon>0$, $N(\varepsilon)$ and a subsequence such that

$$\|\varphi_n(t_n)-\varphi_0\|_{E_0}\geq\varepsilon,$$

and

$$\|\varphi_n(0)-\varphi_0\|_{E_0}<\varepsilon, \text{if } n\geq N.$$

As $\varphi_n(\cdot)\in\mathbf{C}^{\mathrm{loc}}([0,\infty);E_0)$, there exist $s_n\in[0,t_n]$ such that $\|\varphi_n(s_n)-\varphi_0\|_{E_0}=\varepsilon$. But $\{\varphi_n(\cdot)\}\subset U$, so that $\{\varphi_n(s_n)\}\subset A$ is precompact in E_0 and up to a subsequence $\varphi_n(s_n)\to z\in A$ in E_0. Therefore,

$$\|z - \varphi_0\|_{E_0} = \varepsilon. \tag{1.13}$$

Let us consider $v_n(\cdot) = T(s_n)\varphi_n(\cdot) \in K^+$, for which $v_n(0) \to z$ in E_0. Then there exists $\psi(\cdot) \in K^+$ with $\psi(0) = z$ such that $v_n(t) = \varphi_n(s_n + t) \to \psi(t)$ in E_0, for any $t \geq 0$. But for all $t > 0$, we have $\varphi_n(s_n + t) \to \varphi(t)$ in E_0, so $\varphi(t) = \psi(t)$, $\forall\, t > 0$, and due to the continuity of $\varphi(\cdot)$ and $\psi(\cdot)$ on $[0, \infty)$, we have $\varphi(0) = \varphi_0 = \psi(0) = z$. This is a contradiction with (1.13).

Thus, for any sequence $\{\varphi_n(\cdot)\} \subset U$, there exists $\varphi(\cdot) \in K^+$ such that for some subsequence $\varphi_n(\cdot) \to \varphi(\cdot)$ in $\mathbf{C}^{\text{loc}}(\mathbf{R}_+; E_0)$. As $\varphi_n(t) \in A$, $\forall\, t \geq 0$, we have $\varphi(t) \in A$, $\forall\, t \geq 0$.

Let us prove that $\varphi(\cdot) = \Pi_+\psi(\cdot)$, where $\psi(\cdot) \in \mathbf{K}$. Since $\varphi_n(\cdot) \in \Pi_+\mathbf{K}$, there exists $\bar{\varphi}_n(\cdot) \in \mathbf{K}$ such that $\bar{\varphi}_n(t) = \varphi_n(t)$, $\forall\, t \geq 0$. It follows from Lemma 1.8 that $\bar{\varphi}_n(t) \in A$, for all $t \in \mathbf{R}$. Then in a similar way as in Lemma 1.9, one can prove the existence of $\varphi_{-1}(\cdot) \in K^+$ such that, up to a subsequence,

$$\bar{\varphi}_n(t-1) \to \varphi_{-1}(t) \text{ in } \ E_0,\ \forall\, t \geq 0,$$

and, moreover, $\varphi(t) = \varphi_{-1}(t+1)$, $\forall\, t \geq 0$, $\varphi_{-1}(t) \in A$, $\forall\, t \geq 0$. Arguing inductively, we obtain that for any $k \geq 1$, the existence of a function $\varphi_{-k}(\cdot) \in K^+$ such that, up to a subsequence,

$$\bar{\varphi}_n(t-k) \to \varphi_{-k}(t) \text{ in } E_0, \forall\, t \geq 0,$$

and, moreover, $\varphi(t) = \varphi_{-k}(t+k)$, $\forall\, t \geq 0$, $\varphi_{-k}(t) \in A$, $\forall\, t \geq 0$, and

$$\varphi_{-k}(t+k) = \varphi_{-k-1}(t+k+1), \quad \forall\, t \geq -k.$$

So, we can correctly define the following map

$$\psi(t) := \varphi_{-k}(t+k), \text{ if } t \geq -k.$$

We have $\psi(t) \in A$, $\forall\, t \in \mathbf{R}$, so that $\psi(\cdot) \in W$. Also, as for any $h \in \mathbf{R}$, there exists k such that $h + k \geq 0$, we have $s + h \geq -k$, $\forall s \geq 0$, and $\psi_h(s) = \psi(s+h) = \varphi_{-k}(s+h+k)$. Thus,

$$\Pi_+\psi_h(\cdot) = T(h+k)\varphi_{-k}(\cdot) \in K^+$$

and, moreover, $\psi(s) = \varphi(s)$, $\forall s \geq 0$. So, $\varphi(\cdot) = \Pi_+\psi(\cdot)$, where $\psi(\cdot) \in \mathbf{K}$.

The compacity of U in $\mathbf{C}^{\text{loc}}(\mathbf{R}_+; E_0)$ is then proved. □

We shall prove now that U is an attracting set.

Lemma 1.11. *For any $B \subset K^+$, bounded in $L_\infty(\mathbf{R}_+; E)$, we have that (1.4) holds.*

Proof. Let $B \subset K^+$ be bounded in $L_\infty(\mathbf{R}_+; E)$. We shall prove that

$$dist_{C([0,M];E_0)}(\Pi_M T(t) B, \Pi_M U) \to 0, \text{ as } t \to \infty.$$

By contradiction suppose the existence of $\varepsilon > 0$, $M > 0$ and sequences $t_n \to \infty$, $v_n(\cdot) \in B$ such that

$$dist_{C([0,M];E_0)}(\Pi_M v_n(t_n + \cdot), \Pi_M U) \geq \varepsilon, \forall n. \tag{1.14}$$

We note that $B \subset K^+$ is bounded in $L_\infty(\mathbf{R}_+; E)$, so that there exists $c > 0$ such that $\|v(t)\|_E \leq c$, for all $t \geq 0$, $v(\cdot) \in B$. Since $v_n(t_n) \in G(t_n, v_n(0)) \subset G(t_n, B_c)$, where $B_c = \{v \in E : \|v\|_E \leq c\}$, it follows from Theorem 1.9 that up to a subsequence $v_n(t_n) \to z \in A$.

Then arguing exactly in the same way as in the proof of the compacity of U, we obtain that $\varphi_n(\cdot) = v_n(t_n + \cdot)$ converges in $C([0,M]; \mathbf{R}_+)$, for any $M > 0$, to some $\varphi(\cdot) \in K^+$. Moreover, $\varphi(\cdot) = \Pi_+ \psi(\cdot)$ for some $\psi(\cdot) \in \mathbf{K}$. Hence, $\varphi(\cdot) \in U$, which contradicts (1.14). □

As a consequence of Lemmas 1.8–1.11, we obtain that $U \subset K^+$ is compact in $\mathbf{C}^{\mathrm{loc}}(\mathbf{R}_+; E_0)$ and bounded in $L_\infty(\mathbf{R}_+; E)$. Moreover, it is an attracting set in the sense of Definition 1.10 and equality (1.12) holds. Therefore, from Theorem 1.8 and formula (1.5), we have that U is a trajectory attractor in the space of trajectories K^+. The theorem is proved. □

Now, we give an example which the shows the importance of condition (1.11).

Example 1.2. Let us consider again the differential inclusion

$$\dot{u}(t) \leq u^2(t) + 2,$$

with $E = E_0 = \mathbf{R}_+$, but now, we choose K^+ in the following way:

$$\begin{aligned} K^+ = \{u(\cdot) &\text{ is a solution, } u(\cdot) \in L_\infty(\mathbf{R}_+), \\ &u(t) \geq 0, \text{ and for any } \tau \geq 0, \text{ we have} \\ &\qquad u(t+\tau) \leq 1, \ \forall t \geq u(\tau)\}. \end{aligned}$$

In particular, $u(t) \leq 1$, $\forall t \geq u(0)$. Then we have $K^+ \neq \emptyset$, because $u(t) \equiv 0$ belongs to K^+. Moreover, for any $u_0 \in \mathbf{R}_+$, there exists some $u(\cdot) \in K^+$. For example,

$$u(t) = \begin{cases} -t + u_0, & t \in [0, u_0], \\ 0, & t \geq u_0, \end{cases}$$

such that $u(0) = u_0$. Finally, for any $\tau_0 \geq 0$ and $u(\cdot) \in K^+$, we have that $v(\cdot) = u(\cdot + \tau_0)$ is a solution, $v(\cdot) \in L^\infty(\mathbf{R}_+)$, $v(t) \geq 0$, and for all $\tau \geq 0$,

$$v(t+\tau) = u(t + \tau + \tau_0) \leq 1, \ \forall\, t \geq u(\tau + \tau_0) = v(\tau).$$

Then K^+ satisfies conditions (K1)–(K3), and we can construct an m-semiflow G by the formula (1.6). This m-semiflow has the global attractor $A = [0, 1]$. Indeed, $A = [0, 1]$ is compact, and $A \subset G(t, A)$, as for any $u_0 \in A = [0, 1]$, the function $u(t) \equiv u_0$ belongs to K^+. Let $B \subset \mathbf{R}_+$ be an arbitrary bounded set. Then for some $c > 0$, we have $u_0 \le c$, if $u_0 \in B$, and then $u(t) \in [0, 1]$, $\forall\, t \ge u_0$, $\forall\, u(\cdot) \in K^+$ such that $u(0) = u_0$, implies that

$$u(t) \in [0, 1],\ \ \forall t \ge c.$$

Hence,

$$G(t, B) \subset A,\ \forall\, t \ge c.$$

Then from Definition 1.12, we deduce that A is a global attractor for the m-semiflow G.

Condition (1.11) does not hold. Indeed, let us put for $u_0 = 1 \in A$, and for any $N > 1$,

$$u_N(t) = \begin{cases} tg(t + \frac{\pi}{4}), & t \in [0, \frac{\pi}{4} - \frac{1}{N}], \\ -\frac{\alpha_N}{1-\frac{\pi}{4}+\frac{1}{N}} t + \frac{\alpha_N}{1-\frac{\pi}{4}+\frac{1}{N}}, & t \in [\frac{\pi}{4} - \frac{1}{N}, 1], \\ 0, & t \ge 1, \end{cases}$$

where $\alpha_N = tg(\frac{\pi}{2} - \frac{1}{N})$. Then $u_N(0) = 1 \in A$ and $u_N(\cdot)$ is a solution. It is easy to check that for any $\tau \ge 0$, we have $\tau + u_N(\tau) \ge 1$, so $u_N(\cdot) \in K^+$. But

$$u_N\left(\frac{\pi}{4}\right) = \alpha_N \cdot \frac{1 - \frac{\pi}{4}}{1 - \frac{\pi}{4} + \frac{1}{N}} \to \infty,\ \ \text{as } N \to \infty,$$

so that there is no function $u(\cdot) \in K^+$, with $u(0) = 1$, such that for some subsequence $u_N(\frac{\pi}{4}) \to u(\frac{\pi}{4})$, as $N \to \infty$, and then condition (1.11) does not hold.

Let us prove that there is no trajectory attractor in the space of trajectories K^+. For this purpose, it is enough to prove that there exist sequences $t_n \nearrow \infty$, $\{\varphi_n(\cdot)\} \subset K^+$, where the last one is bounded in $L_\infty(\mathbf{R}_+)$, such that the sequence $\{T(t_n)\varphi_n(\cdot)\}$ is not precompact in $\mathbf{C}([0, M])$ for some $M > 0$.

Let us consider for $n \ge 2$ the following periodic functions with period 2:

$$\varphi_n(t) = \begin{cases} 1, & t \in [0, 1], \\ -nt + 1 + n, & t \in [1, 1 + \frac{1}{n}], \\ \frac{n}{n-1} t - \frac{n+1}{n-1}, & t \in [1 + \frac{1}{n}, 2]. \end{cases}$$

Then for any $t \ge 0$, we have $\varphi_n(t) \in [0, 1]$, so that $\{\varphi_n(\cdot)\}$ is bounded in $L_\infty(\mathbf{R}_+)$. Also, $\varphi_n(\cdot)$ is a solution, so it belongs to K^+. Further, for $t_n = 2n$, we have

$$v_n(\cdot) := T(t_n)\varphi_n(\cdot) = \varphi_n(\cdot + 2n) = \varphi_n(\cdot).$$

Let us prove that $\{v_n(\cdot)\}$ is not precompact in $\mathbf{C}([0, 2])$. We note that from

$$|\varphi_n(t) - \varphi_n(s)| = n\,|t - s|\,,$$

for any $t, s \in [1, 1 + \frac{1}{n}]$, it follows that the sequence $\varphi_n(\cdot)$ is not equicontinuous. Then the result follows from the Ascoli–Arzelà compactness criterion.

Therefore, there is no trajectory attractor in the space of trajectories K^+.

1.2.3 Some Model Applications

1.2.3.1 Reaction-Diffusion Equations

Let us consider the problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = a\triangle u(t,x) - f(u(t,x)) + h(x), \\ (t,x) \in (0,+\infty) \times \Omega, \\ u(t,x)|_{x\in\partial\Omega} = 0,\ u(0,x) = u_0(x), \end{cases} \tag{1.15}$$

where $a > 0$ is a constant, $\Omega \subset \mathbf{R}^n$ is a bounded open subset with smooth boundary $\partial\Omega$, $f \in \mathbf{C}^{\text{loc}}(\mathbf{R})$, $h \in L_2(\Omega)$, and the following condition holds: there exist $C_1,\ C_2 > 0,\ \alpha > 0,\ p \geq 2$ such that for all $u \in \mathbf{R}$, we have

$$\begin{aligned} |f(u)| &\leq C_1(1 + |u|^{p-1}), \\ f(u)u &\geq \alpha|u|^p - C_2. \end{aligned} \tag{1.16}$$

We shall define a weak solution of (1.15) as a function from

$$L_2^{loc}(0,+\infty; H_0^1(\Omega)) \cap L_p^{loc}(0,+\infty; L_p(\Omega))$$

which satisfies the equation in (1.15) in the sense of scalar distribution on each interval $(0,T)$. It is known [25] that for every $u_0 \in L_2(\Omega)$, problem (1.15) under conditions (1.16) has at least one weak solution with $u(0) = u_0$, and each weak solution belongs to

$$\mathbf{C}^{\text{loc}}([0,+\infty); L_2(\Omega)).$$

It is proved in [51] (see also [46] for the more general case of reaction-diffusion systems) that for each weak solution of (1.15), the following estimates hold:

$$|u(t)|^2 \leq |u(0)|^2 e^{-\delta t} + \frac{C_3}{\delta}(|h|^2 + 1),\ \forall\, t \geq 0, \tag{1.17}$$

where $|\cdot|$ is the norm in $L_2(\Omega)$ and the positive constants δ, C_3 depend only on the constants of problem (1.15). Moreover, for every sequence of weak solutions $\{u^n(\cdot)\}$, with $u^n(0) \to u_0$ in $L_2(\Omega)$, there exists a subsequence (denoted again by u^n) and a weak solution $u(\cdot)$, $u(0) = u_0$, such that

$$u^n(t) \to u(t) \text{ in } L_2(\Omega),\ \forall\, t \geq 0. \tag{1.18}$$

We denote

$$E = E_0 = L_2(\Omega),$$

and let K^+ be the set of all weak solutions of problem (1.15). From [51] (see also [46]), the m-semiflow G, defined by (1.6), has the global (E, E)-attractor A, which is invariant, stable, and a connected subset of the phase space E.

From (1.18), we obtain that condition (1.11) of Theorem 1.12 holds, so that there exists a trajectory attractor U in the space of trajectories K^+ in the topology $\mathbf{C}^{\text{loc}}(\mathbf{R}_+; L_2(\Omega))$, which is given by formula (1.12). Moreover, by Theorems 1.10 or 1.11, we have $A = U(0)$.

We note that in [22], it is proved the existence of a trajectory attractor in the space of trajectories K^+ with respect to a suitable local weak convergence topology of the space

$$\begin{gathered} L_\infty^{loc}(0, +\infty; E) \cap L_2^{loc}(0, +\infty; H_0^1(\Omega)) \\ \cap L_p^{loc}(0, +\infty; L_p(\Omega)) \\ \cap \{v \mid \partial_t v \in L_q^{loc}(0, +\infty; H^{-r}(\Omega))\}, \\ r = 1/2 - 1/p,\ 1/p + 1/q = 1. \end{gathered} \tag{1.19}$$

Hence, our result concerning the trajectory attractor is sharper. Moreover, we have established the relation $A = U(0)$.

We note that in [46, 49], it is considered a system of reaction-diffusion equations instead of a scalar equation. Under some conditions rather similar to that in (1.16), it is proved the existence of a global compact invariant connected attractor in the phase space $E = E_0 = (L_2(\Omega))^d$. Arguing in the same way as before, we can prove in this more general case the existence of a trajectory attractor in the space of trajectories K^+ in the topology $\mathbf{C}^{\text{loc}}(\mathbf{R}_+; (L_2(\Omega))^d)$, which is given by formula (1.12). Also, $U(0) = A$. Again, this result is sharper than the one in [22].

Also, in [46,49], the result on the existence of the global attractor is applied to the complex Ginzburg–Landau equation and to the Lotka–Volterra system. We obtain that our result about the trajectory attractor is also true for these equations under the same conditions used in that papers.

1.2.3.2 Hyperbolic Equation with Dissipation

Let us consider the problem

$$\begin{cases} u_{tt} + \gamma u_t = \Delta u - f(u) + h(x), \\ (t, x) \in (0, +\infty) \times \Omega, \\ u(t, x)|_{x \in \partial\Omega} = 0,\ u(0, x) = u_0(x), \end{cases} \tag{1.20}$$

where $\gamma > 0$, $\Omega \subset \mathbf{R}^n$ is a bounded open subset with smooth boundary $\partial\Omega$, $n \geq 3$, $f \in \mathbf{C}^{\text{loc}}(\mathbf{R})$, $h \in L_2(\Omega)$, and the following condition holds: there exists $C > 0$

such that for any $u \in \mathbf{R}$,

$$\begin{aligned} |f(u)| &\le C(1+|u|^{\frac{n}{n-2}}), \\ \liminf_{|u|\to\infty} \frac{f(u)}{u} &> -\lambda_1, \end{aligned} \tag{1.21}$$

where $\lambda_1 > 0$ is the first eigenvalue of $-\triangle$ in H_0^1.

We shall denote by $|\cdot|$, $(\cdot,\cdot)$ and $\|\cdot\|$, $((\cdot,\cdot))$ the norm and scalar product in $L_2(\Omega)$ and $H_0^1(\Omega)$, respectively.

The phase space of problem (1.20) is the space

$$E = E_0 = H_0^1(\Omega) \times L_2(\Omega).$$

We shall define a weak solution of (1.20) as a function $\varphi(\cdot) = \begin{pmatrix} u(\cdot) \\ u_t(\cdot) \end{pmatrix} \in L_\infty^{loc}(0,+\infty; E)$, which satisfies the equation in (1.20) in the sense of scalar distribution on each interval $(0,T)$. It is known [10] that for every $\varphi_0 \in E$, problem (1.20) under condition (1.21) has at least one weak solution with $\varphi(0) = \varphi_0$, and each weak solution belongs to

$$\mathbf{C}^{\mathrm{loc}}([0,+\infty); E).$$

Also by Theorem 3.6 in [10], we have that for each weak solution of (1.20) the following estimate holds:

$$\begin{aligned} |u_t(t)|^2 + \|u(t)\|^2 \le C_1(|u_t(0)|^2 + \|u(0)\|^{\frac{2n-2}{n-2}}) \\ +C_2(1+|h|^2), \forall\, t \ge 0, \end{aligned} \tag{1.22}$$

where the positive constants $C_1,\ C_2$ depend only on the constants of problem (1.20). Moreover, for every sequence of weak solutions $\{\varphi^n(\cdot)\}$ with $\varphi^n(0) \to \varphi_0$ in E, there exists some subsequence (denoted again by φ^n) and a weak solution $\varphi(\cdot)$, with $\varphi(0) = \varphi_0$, such that

$$\varphi^n(t) \to \varphi(t) \text{ in } E,\ \forall\, t \ge 0. \tag{1.23}$$

Let K^+ be a set of all weak solutions of problem (1.20). From [10], the m-semiflow G, defined by (1.6), has a global (E,E)-attractor A, which is invariant, stable, and a connected subset of the phase space E.

From (1.23), we obtain that condition (1.11) of Theorem 1.12 holds, so there exists a trajectory attractor U in the space of trajectories K^+ in the topology $\mathbf{C}^{\mathrm{loc}}(\mathbf{R}_+; E)$, which is given by formula (1.12). Also, by Theorem 1.10 or 1.11, we have $A = U(0)$.

We note that in [23], under different conditions than the ones in (1.21), the existence of a trajectory attractor in a specially constructed space of trajectories was proved. If we consider similar arguments as in [23] under condition (1.21), then we can obtain the existence of a trajectory attractor in the space of trajectories

$K^+ = \left\{ \varphi(\cdot) = \begin{pmatrix} u(\cdot) \\ u_t(\cdot) \end{pmatrix} \right\}$ with respect to a suitable local weak-star topology of the space

$$L_\infty^{loc}(0, +\infty; E) \cap \{\partial_t^2 u \in L_\infty^{loc}(0, +\infty; H^{-1}(\Omega))\}.$$

Hence, we have obtained a sharper result concerning the trajectory attractor. Moreover, we have established the relation $A = U(0)$.

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Chapter 2
Auxiliary Properties of Evolution Inclusions Solutions for Earth Data Processing

A great number of collectives of mathematicians, mechanicians, geophysicists (mainly theorists), engineers goes in for qualitative investigation of nonlinear mathematical models of evolution processes and fields of different nature, in particular, problems deal with the dynamics of solutions of non-stationary problems. Far from complete list of results concern the given direction is in works [4, 5, 7, 9–17, 19]. The last results deal with the studying of multivalued, in general case, dynamics of solutions of mathematical models with nonlinear, nonsmooth, nonmonotonic interaction functions as a rule are based on the theory of global and trajectory attractors for m-semiflows of solutions [4,21,24,37]. At that, properties for solutions of considered evolution problem concern with dissipativity of system and closedness (in certain sense) of resolving operator [4, 10, 11, 21, 24, 32, 36, 37]. Note that such properties of solutions for each equation are usually checked separately. At that we succeed to consider problems with linear main part of differential operator appeared in problem [4, 10, 11, 32, 37]. On the other hand, energetic extensions and Nemytskii operators for differential operators appeared in generalized settings of different problems of mathematical physics, problems on a manifold with boundary and without boundary, problems with delay, stochastic partial differential equations, problems with degenerates, as a rule have (as corresponding choice of the phase space) common properties concern growth conditions (usually no more that polynomial), sign conditions, pseudomonotony [14, 16, 19, 22, 25, 41, 42]. In general case as such restrictions for determinative parameters of a problem we succeed to prove only the existence of weak solutions of differential-operator inclusion, but not always this proof is constructive [14, 16, 19, 22, 25, 41, 42]. Hence, the problem of the existence of trajectory and global attractors and investigation of their structure for weak solutions of differential-operator equations in infinite-dimensional spaces with interaction functions of pseudomonotone type is actual one. Here we consider some additional properties of solutions for the first and second order autonomous evolution inclusions with pointwice pseudomonotone multivalued maps. This properties are connected with dissipation and closuredness of graph for resolving operator. The results of this chapter are borrowed from [6, 8, 13, 15, 18, 21, 23, 24, 28, 29, 40, 43].

M.Z. Zgurovsky et al., *Evolution Inclusions and Variation Inequalities for Earth Data Processing III*, Advances in Mechanics and Mathematics 27,
DOI 10.1007/978-3-642-28512-7_2, © Springer-Verlag Berlin Heidelberg 2012

2.1 Preliminaries

At first let us consider constructions, presented in [41, 42].

At an analysis and control of different geophysical and socio-economical processes it is often appears such problem: at a mathematical modelling of effects related to friction and viscosity, quantum effects, a description of different nature waves the existing "gap" between rather high degree of the mathematical theory of analysis and control for non-linear processes and fields and practice of its using in applied scientific investigations make us require rather stringent conditions for interaction functions. These conditions related to linearity, monotony, smoothness, continuity and can substantially have an influence on the adequacy of mathematical model. Let us consider for example some diffusion process. Its mathematical model has the next form:

$$\begin{cases} y_t - \Delta y + f(y) = g(t,x) \quad in \quad \Omega \times (\tau; T), \\ y\big|_{\partial\Omega} = 0, \\ y\big|_{\tau=T} = y_0, \end{cases} \tag{2.1}$$

here $n \geq 2$, $\Omega \subset \mathbf{R}^n$ is a bounded domain with a rather smooth boundary, $-\infty < \tau < T < +\infty$, $g : \Omega \times (\tau; T) \to \mathbf{R}$, $y_0 : \Omega \to \mathbf{R}$ are rather regular functions, $f : \mathbf{R} \to \mathbf{R}$ is an interaction function, $y : \Omega \times (\tau; T) \to \mathbf{R}$ is an unknown function. It is well known that if f is a rather smooth function and satisfies for example the next condition of no more than polynomial growth:

$$\exists p > 1, \quad \exists c > 0 : \quad |f(s)| \leq c(1 + |s|^{p-1}) \quad \forall s \in \mathbf{R}, \tag{2.2}$$

then problem (2.1) has a unique rather regular solution. Let us consider the case when f is continuous and initial data and external forces are nonregular (for example $y_0 \in L_2(\Omega)$, $g \in L_2(\Omega \times (\tau; T))$). Then, as a rule, we consider the generalized setting of problem (2.1):

$$\begin{cases} y'(t) + A(y(t)) + B(y(t)) = g(t) \quad \text{for a.e.} \quad t \in (\tau; T), \\ y(\tau) = y_0, \end{cases} \tag{2.3}$$

here $A : V_1 \to V_1^*$ is an energetic extension of operator "$-\Delta$", $B : V_2 \to V_2^*$ is the Nemytskii operator for F, $V_1 = H_0^1(\Omega)$ is a real Sobolev space, $V_2 = L_p(\Omega)$, $V_1^* = H^{-1}(\Omega)$, $V_2^* = L_q(\Omega)$, q is the conjugated index, y' is a derivative of an element $y \in L_2(\tau, T; V_1) \cap L_p(\tau, T; V_2)$ and it is considered in the sense of the space $\mathscr{D}^*([\tau; T], V_1^* + V_2^*)$.

A solution of problem (2.3) in the class $W = \{y \in L_2(\tau, T; V_1) \cap L_p(\tau, T; V_2) \big| y' \in L_2(\tau, T; V_1^*) + L_q(\tau, T; V_2^*)\}$ refers to be the generalized solution of problem (2.1).

To prove the existence of solutions for problem (2.1) as a rule we need to add supplementary "signed condition" for an interaction function f, for example,

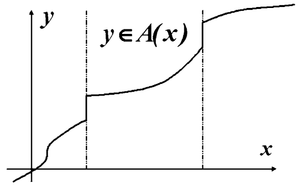

Fig. 2.1 The monotone multivalued map

$$\exists \alpha, \beta > 0 : \quad f(s)s \geq \alpha|s|^p - \beta \quad \forall s \in \mathbf{R}. \tag{2.4}$$

But we do not succeed in proving the uniqueness of the solution of such problem in the general case. Note that technical condition (2.4) provides a dissipation too. We remark also that different conditions for parameters of problem (2.1) provide corresponding conditions for generated mappings A and B.

Problem (2.3) is usually investigated in more general case:

$$\begin{cases} y' + \mathscr{A}(y) = g, \\ y(\tau) = y_0, \end{cases} \tag{2.5}$$

here $\mathscr{A} : X \to X^*$ is the Nemytskii operator for $A + B$,

$$\mathscr{A}(y)(t) = A(y(t)) + B(y(t)) \quad \text{for a.e.} \quad t \in (\tau; T),\ y \in X,$$

$$X = L_2(\tau, T; V_1) \cap L_p(\tau, T; V_2), \quad X^* = L_2(\tau, T; V_1^*) + L_q(\tau, T; V_2^*).$$

Solutions of problem (2.5) are also searched in the class $W = \{y \in X \,\big|\, y' \in X^*\}$.

In cases when the continuity of the interaction function f have an influence on the adequacy of mathematical model fundamentally then problem (2.1) is reduced to such problem (Fig. 2.1):

$$\begin{cases} y_t - \Delta y + F(y) \ni g(x,t) \quad \text{in } Q = \Omega \times (\tau; T), \\ y\big|_{\partial\Omega} = 0, \\ y\big|_{\tau=T} = y_0, \end{cases} \tag{2.6}$$

here

$$F(s) = [\underline{f}(s), \overline{f}(s)], \quad \underline{f}(s) = \varliminf_{t\to s} f(t), \quad \overline{f}(s) = \varlimsup_{t\to s} f(t), \quad s \in \mathbf{R},$$

$$[a,b] = \{\alpha a + (1-\alpha)b \,\big|\, \alpha \in [0,1]\},$$

$-\infty < a < b < +\infty$.

A solution of such differential-operator inclusion

$$\begin{cases} y' + \mathscr{A}(y) \ni g, \\ y(\tau) = y_0, \end{cases} \tag{2.7}$$

is usually thought to be the generalized solution of problem (2.6). Here $\mathscr{A} : X \rightrightarrows X^*$,

$$\mathscr{A}(u) = \{p \in X^* \big| p(t) \in A(u(t)) + B(u(t)), \text{ for a.e. } t \in (\tau; T)\}, \; u \in X,$$

$A : V_1 \to V_1^*$ is the energetic extension of "$-\Delta$" in $H_0^1(\Omega)$, $B : V_2 \to C_v(V_2^*)$ is the Nemytskii operator for F:

$$B(v) = \{z \in V_2^* \big| z(x) \in F(v(x)) \text{ for a.e. } x \in \Omega\}, \quad v \in V_2.$$

Taking into account all variety of classes of mathematical models for different nature geophysical processes and fields we propose rather general approach to investigation of them in this book. Further we will study classes of mathematical models in terms of general properties of generated mappings like $\mathscr{A}$.

Let us consider some denotations and results, that we will use in this book. Let X be a Banach space, X^* be its topologically adjoint,

$$\langle \cdot, \cdot \rangle_X : X^* \times X \to \mathbf{R}$$

be the canonical duality between X and X^*, 2^{X^*} be a family of all subsets of the space X^*, let $A : X \to 2^{X^*}$ be the multivalued map,

$$\text{graph}A = \{(\xi; y) \in X^* \times X \,|\xi \in A(y)\},$$
$$\text{Dom}A = \{y \in X \,|A(y) \neq \emptyset\}.$$

The multivalued map A is called strict if $\text{Dom}A = X$. Together with every multivalued map A we consider its upper

$$[A(y), \xi]_+ = \sup_{d \in A(y)} \langle d, \xi \rangle_X$$

and lower

$$[A(y), \xi]_- = \inf_{d \in A(y)} \langle d, \xi \rangle_X$$

support functions, where $y, \xi \in X$. Let also

$$\|A(y)\|_+ = \sup_{d \in A(y)} \|d\|_{X^*}, \quad \|A(y)\|_- = \inf_{d \in A(y)} \|d\|_{X^*}, \quad \|\emptyset\|_+ = \|\emptyset\|_- = 0.$$

For arbitrary sets $C, D \in 2^{X^*}$ we set

$$\text{dist}(C, D) = \sup_{e \in C} \inf_{d \in D} \|e - d\|_{X^*}, \quad d_H(C, D) = \max\{\text{dist}(C, D), \text{dist}(D, C)\}.$$

Then, obviously,

$$\|A(y)\|_+ = d_H(A(y), 0) = \text{dist}(A(y), 0), \quad \|A(y)\|_- = \text{dist}(0, A(y)).$$

Together with the operator $A : X \to 2^{X^*}$ let us consider the following maps

$$\mathrm{co}A : X \to 2^{X^*} \quad \text{and} \quad \overset{*}{\overline{\mathrm{co}}}\, A : X \to 2^{X^*},$$

defined by relations

$$(\mathrm{co}A)(y) = \mathrm{co}(A(y)) \quad \text{and} \quad (\overset{*}{\overline{\mathrm{co}}}\, A(y)) = \overset{*}{\overline{\mathrm{co}}}\, (A(y))$$

respectively, where $\overset{*}{\overline{\mathrm{co}}}\, (A(y))$ is the weak star closure of the convex hull $\mathrm{co}(A(y))$ for the set $A(y)$ in the space X^*. Besides for every $G \subset X$

$$(\mathrm{co}A)(G) = \bigcup_{y \in G} (\mathrm{co}A)(y), \quad (\overset{*}{\overline{\mathrm{co}}}\, A)(G) = \bigcup_{y \in G} (\overset{*}{\overline{\mathrm{co}}}\, A)(y).$$

Further we will denote the strong, weak and weak star convergence by $\to$, $\overset{w}{\to}$, $\overset{*}{\to}$ or $\to$, $\rightharpoonup$, $\rightharpoonup$ respectively. As $C_v(X^*)$ we consider the family of all nonempty convex closed bounded subsets from X^*.

Proposition 2.1. *[41, Proposition 1.2.1] Let* $A, B, C : X \rightrightarrows X^*$. *Then for all* $y, v, v_1, v_2 \in X$ *the following statements take place:*

1. *The functional* $X \ni u \to [A(y), u]_+$ *is convex, positively homogeneous and lower semicontinuous;*
2. $[A(y), v_1 + v_2]_+ \le [A(y), v_1]_+ + [A(y), v_2]_+$,
   $[A(y), v_1 + v_2]_- \ge [A(y), v_1]_- + [A(y), v_2]_-$,
   $[A(y), v_1 + v_2]_+ \ge [A(y), v_1]_+ + [A(y), v_2]_-$,
   $[A(y), v_1 + v_2]_- \le [A(y), v_1]_+ + [A(y), v_2]_-$;
3. $[A(y) + B(y), v]_+ = [A(y), v]_+ + [B(y), v]_+$,
   $[A(y) + B(y), v]_- = [A(y), v]_- + [B(y), v]_-$;
4. $[A(y), v]_+ \le \|A(y)\|_+ \|v\|_X$,
   $[A(y), v]_- \le \|A(y)\|_- \|v\|_X$;
5. $\| \overset{*}{\overline{\mathrm{co}}}\, A(y) \|_+ = \|A(y)\|_+$, $\| \overset{*}{\overline{\mathrm{co}}}\, A(y) \|_- = \|A(y)\|_-$,
   $[A(y), v]_+ = \left[\overset{*}{\overline{\mathrm{co}}}\, A(y), v \right]_+$, $[A(y), v]_- = \left[\overset{*}{\overline{\mathrm{co}}}\, A(y), v \right]_-$;
6. $\|A(y) - B(y)\|_+ \ge | \, \|A(y)\|_+ - \|B(y)\|_- \, |$,
   $\|A(y) - B(y)\|_- \ge \|A(y)\|_- - \|B(y)\|_+$;
7. $d \in \overset{*}{\overline{\mathrm{co}}}\, A(y) \quad \Leftrightarrow \quad \forall \omega \in X \ [A(y), \omega]_+ \ge \langle d, w \rangle_X$;
8. $d_H(A(y), B(y)) \ge |\|A(y)\|_+ - \|B(y)\|_+|$,
   $d_H(A(y), B(y)) \ge |\|A(y)\|_+ - \|B(y)\|_+|$,
   where d_H *is Hausdorff metric;*
9. $\mathrm{dist}(A(y) + B(y), C(y)) \le \mathrm{dist}(A(y), C(y)) + \mathrm{dist}(B(y), 0)$,
   $\mathrm{dist}(C(y), A(y) + B(y)) \le \mathrm{dist}(C(y), A(y)) + \mathrm{dist}(0, B(y))$,
   $d_H(A(y) + B(y), C(y)) \le d_H(A(y), C(y)) + d_H(B(y), 0)$;

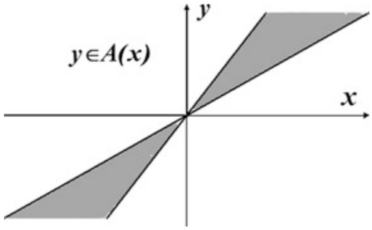

Fig. 2.2 The "−"-coercive multivalued map

10. For any $D \subset X^*$ *and bounded* $E \in C_v(X^*)$

$$\operatorname{dist}(D, E) = \operatorname{dist}(\overset{*}{\overline{\text{co}}}\, D, E).$$

Proposition 2.2. *[41, Proposition 1.2.2] The inclusion* $d \in \overset{*}{\overline{\text{co}}}\, A(y)$ *holds true if and only if one of the following relations takes place (Fig. 2.2):*

$$\begin{aligned} \textit{either} \quad & [A(y), v]_+ \geq \langle d, v\rangle_X \quad \forall v \in X, \\ \textit{or} \quad & [A(y), v]_- \leq \langle d, v\rangle_X \quad \forall v \in X. \end{aligned}$$

Proposition 2.3. *[41, Proposition 1.2.3] Let* $D \subset X$ *and* $a(\cdot,\cdot) : D \times X \to \mathbf{R}$*. For each* $y \in D$ *the functional* $X \ni w \mapsto a(y, w)$ *is positively homogeneous, convex and lower semicontinuous if and only if there exists the multivalued map* $A : X \to 2^{X^*}$ *such that* $D(A) = D$ *and*

$$a(y, w) = [A(y), w]_+ \qquad \forall y \in D(A),\ w \in X.$$

Proposition 2.4. *[41, Proposition 1.2.4] The functional* $\|\cdot\|_+ : C_v(X^*) \to \mathbf{R}_+$ *satisfies the following properties:*

1. $\{\bar{0}\} = A \quad \Leftrightarrow \quad \|A\|_+ = 0$,
2. $\|\alpha A\|_+ = |\alpha|\|A\|_+, \quad \forall \alpha \in \mathbf{R},\ A \in C_v(X^*)$,
3. $\|A + B\|_+ \leq \|A\|_+ + \|B\|_+ \quad \forall A, B \in C_v(X^*)$.

Proposition 2.5. *[41, Proposition 1.2.5] The functional* $\|\cdot\|_- : C_v(X^*) \to \mathbf{R}_+$ *satisfies the following properties:*

1. $\bar{0} \in A \quad \Leftrightarrow \quad \|A\|_- = 0$,
2. $\|\alpha A\|_- = |\alpha|\|A\|_-, \quad \forall \alpha \in \mathbf{R},\ A \in C_v(X^*)$,
3. $\|A + B\|_- \leq \|A\|_- + \|B\|_- \quad \forall A, B \in C_v(X^*)$.

Let us remark that any multivalued map $A : X \to 2^{X^*}$, naturally generates *upper* and, accordingly, *lower form*:

$$[A(y), \omega]_+ = \sup_{d \in A(y)} \langle d, w\rangle_X, \qquad [A(y), \omega]_- = \inf_{d \in A(y)} \langle d, w\rangle_X, \quad y, \omega \in X.$$

Thus, together with the classical coercivity condition for operator A:

$$\frac{\langle A(y), y\rangle_X}{\|y\|_X} \to +\infty, \qquad \text{as} \qquad \|y\|_X \to +\infty,$$

which ensures the important a priori estimations, arises $+$-coercivity (and, accordingly, $-$-coercivity):

$$\frac{[A(y), y]_{+(-)}}{\|y\|_X} \to +\infty, \qquad \text{as} \qquad \|y\|_X \to +\infty.$$

$+$-coercivity is much weaker condition than $-$-coercivity.

2.2 Pointwise Pseudomonotone Maps

In this section we consider Nemytskii operator properties for classes of pseudomonotone multivalued maps, considered in [20] (see paper and references therein). This properties we obtain, analyzing Theorem proofs from [20]. At that we consider weaker properties for operators connected with measurability and obtain stronger results, that we use in further sections.

For evolution triple (V, H, V^*),[1] $p > 1$ we consider a multivalued (in the general case) map $A : V \overset{\rightarrow}{\rightarrow} V^*$. We suppose

$(A1)$ $v \to A(v)$ is a pseudomonotone map such that

(a) $A(u) \in C_v(V^*)$ $\forall u \in V$, i.e. the set $A(u)$ is a nonempty, closed and convex one for all $u \in V$;

(b) If $u_j \to u$ weakly in V and $d_j \in A(u_j)$ is such that

$$\overline{\lim_{j\to+\infty}} \langle d_j, u_j - u\rangle_V \le 0,$$

then

$$\underline{\lim_{j\to+\infty}} \langle d_j, u_j - \omega\rangle_V \ge [A(u), u - \omega]_- \ \forall \omega \in V.$$

$(A2)$ $\exists c_1 > 0$:

$$\|A(u)\|_+ \le c_1(1 + \|u\|_V^{p-1}) \ \forall u \in V;$$

[1] That is, V is a real reflexive separable Banach space embedded into a real Hilbert space H continuously and densely, H is identified with its conjugated space H^*, V^* is a dual space to V. So, we have such chain of continuous and dense embeddings: $V \subset H \equiv H^* \subset V^*$ (see, for example, [42]).

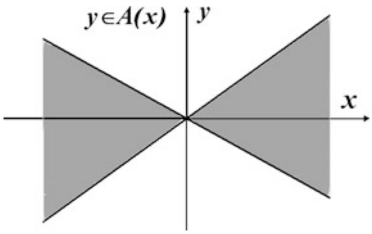

Fig. 2.3 The "+"-coercive multivalued map, but not "−"-coercive

$(A3)$ $\exists c_2, c_3 > 0:$

$$[A(u), u]_- \geq c_2 \|u\|_V^p - c_3 \ \forall u \in V.$$

We consider a reflexive separable Banach space V_σ such that $V_\sigma \subset V$ with dense and continuous embedding. Therefore, we have the chain of continuous and dense embeddings (Fig. 2.3):

$$V_\sigma \subset V \subset H \equiv H^* \subset V^* \subset V_\sigma^*,$$

where V_σ^* is dual space to V_σ. Let us set: $S = [\tau, T], -\infty < \tau < T < +\infty, q > 1:$ $\frac{1}{p} + \frac{1}{q} = 1,$

$$X = L_p(S; V),\ X^* = L_q(S; V^*),\ X_\sigma = L_p(S; V_\sigma),\ X_\sigma^* = L_q(S; V_\sigma^*),$$
$$W = \{y \in X \mid y' \in X^*\},\ W_\sigma = \{y \in X \mid y' \in X_\sigma^*\}.$$

Lemma 2.1. *Under above conditions for any $y \in X$*

$$\hat{A}(y) = \{g \in X^* |\ g(t) \in A(y(t)) \textit{ for a.e. } t \in S\} \neq \emptyset.$$

Moreover,

$$\exists C_1 > 0: \qquad \|\hat{A}(y)\|_+ \leq c_1(1 + \|y\|_X^{p-1}) \ \forall y \in X; \tag{2.8}$$

$$\exists C_2, C_3 > 0: \ [\hat{A}(y), y]_- \geq C_2 \|y\|_X^p - C_3 \ \forall y \in X. \tag{2.9}$$

Proof. Let $y \in X$. Then there exists a sequence of "step functions" [16, Chap. IV] $\{y_n\}_{n \geq 1} \subset X$ such that

$$y_n \to y \text{ in } X, \tag{2.10}$$

$$\text{for a.e. } t \in S \ \ y_n(t) \to y(t) \text{ in } V,\ n \to +\infty. \tag{2.11}$$

We remark that

$$y_n(t) = a_{k,n} \text{ for a.e. } t \in A_{k,n},$$

where $n \geq 1$, $k = 1, \ldots, m_n$, $m_n \in \mathbf{N}$, $A_{k,n}$ is measurable set, $A_{k,n} \cap A_{j,n} = \emptyset$, $k \neq j$, $\bigcup\limits_{k=1}^{m_n} A_{k,n} = S$, $a_{k,n} \in V$.

Let for $n \geq 1$, $k = 1, \ldots, m_n$ $d_{k,n} \in A(a_{k,n})$ be an arbitrary. For any $n \geq 1$ we consider a "step function" $d_n \in X^*$ such that $d_n(t) = d_{k,n}$, for a.e. $t \in A_{k,n}$, $k = \overline{1, m_n}$.

Thus $\forall n \geq 1$ for a.e. $t \in S$ $d_n(t) \in A(y_n(t))$. In virtue of Condition $(A2)$ and (2.10) we obtain that up to a subsequence $\{d_{n_k}\}_{k\geq 1} \subset \{d_n\}_{n\geq 1}$ for some $d \in X^*$ the next convergence is fulfilled:

$$d_{n_k} \to d \text{ weakly in } X^*, \; k \to +\infty. \tag{2.12}$$

To finish the proof of Lemma 2.1 it is sufficiently to show that $d \in \hat{A}(y)$. From (2.11) and Condition $(A2)$ it follows that for a.e. $t \in S$

$$\langle d_n(t), y_n(t) - y(t)\rangle_V \to 0, \; n \to +\infty. \tag{2.13}$$

As V is separable Banach space then there exists a countable dense system of vectors $\{v_j\}_{j\geq 1} \subset V$.

We finish the proof into several steps.

Step 1. In virtue of the pseudomonotony of A, from (2.11), (2.13) it follows that

$$\text{for a.e. } t \in S \; \forall j \geq 1 \; \varliminf_{n\to+\infty} \langle d_n(t), y_n(t) - \omega_j\rangle_V$$
$$= \varliminf_{n\to+\infty} \langle d_n(t), y(t) - \omega_j\rangle_V \geq [A(y(t)), y(t) - \omega_j]_-. \tag{2.14}$$

Step 2. Due to Conditions $(A2)$ and $(A3)$ it follows that $\forall n, j \geq 1$, for a.e. $s \in S$

$$\langle d_n(s), y_n(s) - \omega_j\rangle_V \geq c_2\|y_n(s)\|_V^p - c_3 - c_1(1 + \|y_n(s)\|_V^{p-1})\|\omega_j\|_V.$$

Now using Young's inequality, we can obtain

$$c_1\|y_n(s)\|_V^{p-1}\|\omega_j\|_V \leq c_2\|y_n(s)\|_V^p + c_2^{\frac{-p}{q}}\|\omega_j\|_V^p c_1^p p^{-1} q^{\frac{-p}{q}}.$$

Letting

$$c_{4,j} = c_1\|\omega_j\|_V + c_2^{\frac{-p}{q}}\|\omega_j\|_V^p c_1^p p^{-1} q^{\frac{-p}{q}} + c_3 > 0,$$

we finally get

$$\forall n, j \geq 1, \text{ for a.e. } t \in S \; \langle d_n(s), y_n(s) - \omega_j\rangle_V \geq -c_{4,j}. \tag{2.15}$$

Step 3. From (2.10) and (2.12) we have that $\forall t_1, t_2 \in S$, $t_1 < t_2$,

$$\int_{t_1}^{t_2} \langle d_{n_k}(s), y_{n_k}(s) - \omega_j \rangle_V ds \to \int_{t_1}^{t_2} \langle d(s), y(s) - \omega_j \rangle_V ds. \tag{2.16}$$

Step 4. In virtue of (2.10), (2.12), (2.15), (2.16) and Fatou's lemma $\forall j \geq 1$, $\forall t \in S$, $\forall h > 0 : t + h \in S$, we obtain

$$\int_{t}^{t+h} \langle d(s), y(s) - \omega_j \rangle_V ds = \lim_{k \to +\infty} \int_{t}^{t+h} \langle d_{n_k}(s), y_{n_k}(s) - \omega_j \rangle_V ds$$

$$\geq \underline{\lim}_{k \to +\infty} \int_{t}^{t+h} [\langle d_{n_k}(s), y_{n_k}(s) - \omega_j \rangle_V + c_{4,j}] ds - c_{4,j} h$$

$$\geq \int_{t}^{t+h} \underline{\lim}_{k \to +\infty} [\langle d_{n_k}(s), y_{n_k}(s) - \omega_j \rangle_V + c_{4,j}] ds - c_{4,j} h$$

$$= \int_{t}^{t+h} \underline{\lim}_{k \to +\infty} \langle d_{n_k}(s), y(s) - \omega_j \rangle_V ds.$$

Because of $\forall \varphi \in L_1(S)$

$$\frac{1}{h} \int_0^h \varphi(s + \cdot) ds \to \varphi(\cdot) \text{ in } L_1(S), \ h \searrow 0,$$

we have:

$$\text{for a.e. } t \in S, \ \forall j \geq 1, \ \langle d(t), y(t) - \omega_j \rangle_V \geq \underline{\lim}_{k \to +\infty} \langle d_{n_k}(t), y(t) - \omega_j \rangle_V$$

$$\geq [A(y(t)), y(t) - \omega_j]_-.$$

Step 5. In virtue of $\{y(t) - \omega_j\}_{j \geq 1}$ is dense in V for a.e. $t \in S$ we finally obtain that $d(t) \in A(y(t))$ for a.e. $t \in S$, i.e. $d \in \hat{A}(y)$.

The proof of (2.8), (2.9) is trivial [20].

Lemma 2.2. *Under the above listed conditions, if* $y_n \to y$ *weakly in* W_σ, $\{d_n\}_{n \geq 1} \subset X^* : d_n(t) \in A(y_n(t))$ *for a.e.* $t \in S$, $\forall n \geq 1$, *and*

$$\overline{\lim}_{n \to +\infty} \langle d_n, y_n - y \rangle_X \leq 0,$$

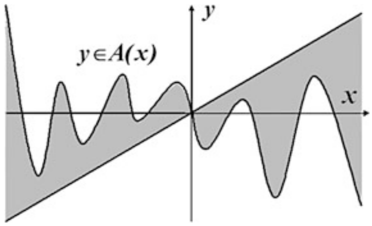

Fig. 2.4 The weakly "−"-coercive multivalued map

we have

$$\lim_{n\to+\infty}\int_S |\langle d_n(t), y_n(t)-y(t)\rangle_V|\,dt = 0. \tag{2.17}$$

Proof. We define $\hat{A}(y) = \{g \in X^* \mid g(t) \in A(y(t))$ for a.e. $t \in S\}$, $y \in X$. From Lemma 2.1 the set $\hat{A}(y)$ is nonempty. It is clear that $\hat{A}(y)$ is a closed and convex set, i.e. $\hat{A}(y) : X \to C_v(X^*)$ (Fig. 2.4).

Let $y_n \to y$ in W_σ, $d_n \in \hat{A}(y_n)$ $\forall n \geq 1$, and we suppose that

$$\overline{\lim_{n\to+\infty}}\langle d_n, y_n - y\rangle_X \leq 0. \tag{2.18}$$

First we prove (2.17). We note that there is a set of measure zero, $\Sigma_1 \subset S$ such that for $t \notin \Sigma_1$, we have that

$$d_n(t) \in A(y_n(t)) \text{ for all } n \geq 1.$$

Similarly to [20, p. 7] we verify the following claim.
Claim: Let $y_n \to y$ weakly in W_σ and let $t \notin \Sigma_1$. Then

$$\underline{\lim_{n\to+\infty}}\ \langle d_n(t), y_n(t) - y(t)\rangle_V \geq 0.$$

Proof of the claim. Fix $t \notin \Sigma_1$ and suppose to the contrary that

$$\underline{\lim_{n\to+\infty}}\ \langle d_n(t), y_n(t) - y(t)\rangle_V < 0. \tag{2.19}$$

Then up to a subsequence $\{d_{n_k}, y_{n_k}\}_{k\geq1} \subset \{d_n, y_n\}_{n\geq1}$ we have

$$\lim_{k\to+\infty}\langle d_{n_k}(t), y_{n_k}(t) - y(t)\rangle_V = \underline{\lim_{n\to+\infty}}\ \langle d_n(t), y_n(t) - y(t)\rangle_V < 0. \tag{2.20}$$

Therefore, for all rather large k, Conditions $(A2)$ and $(A3)$ implies

$$c_2\|y_{n_k}(t)\|_V^p - c_3 \leq \|A(y_{n_k}(t))\|_+\|y(t)\|_V \leq c_1(1+\|y_{n_k}(t)\|_V^{p-1})\|y(t)\|_V.$$

which implies $\{\|y_{n_k}(t)\|_V\}_{k\geq 1}$ and consequently $\{\|d_{n_k}(t)\|_{V^*}\}_{k\geq 1}$ are bounded sequences. $\{\|d_{n_k}(t)\|_{V^*}\}_{k\geq 1}$ is bounded one independently on n_k in virtue of the assumption that $A : V \to C_v(V^*)$ is bounded map and we just showed that $\{\|y_{n_k}(t)\|_V\}_{k\geq 1}$ is bounded sequence. In virtue of the continuity of embedding $W_\sigma \subset C(S; V_\sigma^*)$ we obtain that $y_{n_k}(t) \to y(t)$ weakly in V_σ^* and in virtue of boundedness of $\{y_{n_k}(t)\}_{k\geq 1}$ in V we finally have

$$\forall t \in S \setminus \Sigma_1 \ y_{n_k}(t) \to y(t) \text{ weakly in } V, \ k \to +\infty. \tag{2.21}$$

The pseudomonotony condition for A, (2.19)–(2.21) implies that

$$\varliminf_{n\to+\infty} \langle d_n(t), y_n(t) - y(t)\rangle_V \geq [A(y(t)), y(t) - y(t)]_- \\ = 0 > \varlimsup_{n\to+\infty} \langle d_n(t), y_n(t) - y(t)\rangle_V.$$

We obtain a contradiction.

The claim is proved.

Now we continue the proof of the lemma. It follows from the claim that for a.e. $t \in S$, in fact for any $t \notin \Sigma_1$, we have

$$\varliminf_{n\to+\infty} \langle d_n(t), y_n(t) - y(t)\rangle_V \geq 0. \tag{2.22}$$

Now also from the "–"-coercivity condition, $(A3)$, if $\omega \in X$

$$\langle d_n(t), y_n(t) - \omega(t)\rangle_V \geq c_2\|y_n(t)\|_V^p - c_3 - c_1(1 + \|y_n(t)\|_V^{p-1})\|\omega(t)\|_V \\ \text{for a.e. } t \in S \setminus \Sigma_1.$$

Using $p - 1 = \frac{p}{q}$, the right side of the above inequality equals to

$$c_2\|y_n(t)\|_V^p - c_3 - c_1\|y_n(t)\|_V^{\frac{p}{q}}\|\omega(t)\|_V - c_1\|\omega(t)\|_V.$$

Now using Young's inequality, we can obtain a constant $c(c_1, c_2)$ depending on c_1, c_2 such that

$$c_1\|y_n(t)\|_V^{\frac{p}{q}}\|\omega(t)\|_V \leq \frac{c_2}{2}\|y_n(t)\|_V^p + \|\omega(t)\|_V^p \cdot c(c_1, c_2).$$

Letting $c_4 = \max\{c_3 + \frac{c_1}{q}; c(c_1, c_2) + \frac{c_1}{p}\}$ it follows that

$$\langle d_n(t), y_n(t) - \omega(t)\rangle_V \geq -c_4(1 + \|\omega(t)\|_V^p) \text{ for a.e. } t \in S. \tag{2.23}$$

Letting $\omega = y$, we can use Fatou's lemma and we obtain

$$\varliminf_{n\to+\infty} \int_0^T [\langle d_n(t), y_n(t) - y(t)\rangle_V + c_4(1 + \|y(t)\|_V^p)]dt$$

$$\geq \int_0^T \varliminf_{n\to+\infty} [\langle d_n(t), y_n(t) - y(t)\rangle_V + c_4(1+\|y(t)\|_V^p)]dt \geq c_4 \int_0^T (1+\|y(t)\|_V^p)dt.$$

Therefore,

$$0 \geq \varlimsup_{n\to+\infty} \langle d_n, y_n - y\rangle_X \geq \varliminf_{n\to+\infty} \int_S \langle d_n(t), y_n(t) - y(t)\rangle_V dt$$

$$= \varliminf_{n\to+\infty} \langle d_n, y_n - y\rangle_X \geq \int_S \varliminf_{n\to+\infty} \langle d_n(t), y_n(t) - y(t)\rangle_V dt = 0,$$

showing that

$$\lim_{n\to+\infty} \langle d_n, y_n - y\rangle_X = 0. \tag{2.24}$$

From (2.23),

$$\forall n \geq 1 \ \forall t \notin \Sigma_1 \ 0 \leq \langle d_n(t), y_n(t) - y(t)\rangle_V^- \leq c_4(1 + \|y(t)\|_V^p),$$

where $a^- = \max\{0, -a\}$, for $a \in \mathbf{R}$. Thanks to (2.22) we know that for a.e. t, $\langle d_n(t), y_n(t) - y(t)\rangle_V \geq -\varepsilon$ for all rather large n. Therefore, for such n, $\langle d_n(t), y_n(t) - y(t)\rangle_V^- \leq \varepsilon$, if $\langle d_n(t), y_n(t) - y(t)\rangle_V < 0$ and $\langle d_n(t), y_n(t) - y(t)\rangle_V^- = 0$, if $\langle d_n(t), y_n(t) - y(t)\rangle_V \geq 0$. Therefore, $\lim_{n\to+\infty} \langle d_n(t), y_n(t) - y(t)\rangle_V^- = 0$ and we can apply the dominated convergence theorem and conclude that

$$\lim_{n\to+\infty} \int_S \langle d_n(t), y_n(t) - y(t)\rangle_V^- = \int_S \lim_{n\to+\infty} \langle d_n(t), y_n(t) - y(t)\rangle_V^- dt = 0$$

from (2.22). Now by (2.24) and the above equation we have

$$\lim_{n\to+\infty} \int_S \langle d_n(t), y_n(t) - y(t)\rangle_V^+ dt$$

$$= \lim_{n\to+\infty} \int_0^T [\langle d_n(t), y_n(t) - y(t)\rangle_V + \langle d_n(t), y_n(t) - y(t)\rangle_V^-]dt$$

$$= \lim_{n\to+\infty} \langle d_n, y_n - y\rangle_X = 0.$$

Therefore,

$$\lim_{n\to+\infty}\int_S |\langle d_n(t), y_n(t) - y(t)\rangle_V|dt = 0.$$

The lemma is proved.

Lemma 2.3. *Under the conditions of Lemma 2.2 we additionally have that up to a subsequence* $\{y_{n_k}, d_{n_k}\}_{k\geq 1} \subset \{y_n, d_n\}_{n\geq 1}$ *for a.e.* $t \in S$ $y_{n_k}(t) \to y(t)$ *weakly in* V, *and* $\langle d_{n_k}(t), y_{n_k}(t) - y(t)\rangle_V \to 0$, $k \to +\infty$.

Proof. Let $y_n \to y$ weakly in W_σ, $\{d_n\}_{n\geq 1} \subset X_\sigma^* : d_n(t) \in A(y_n(t))$ for a.e. $t \in S$ $\forall n \geq 1$ and

$$\overline{\lim_{n\to+\infty}} \langle d_n), y_n - y\rangle_X \leq 0.$$

In virtue of Lemma 2.2 we obtain

$$\lim_{n\to+\infty}\int_S |\langle d_n(t), y_n(t) - y(t)\rangle_V|dt = 0. \tag{2.25}$$

Due to the continuous embedding $W_\sigma \subset C(S; V_\sigma^*)$ we have

$$\forall t \in S \;\; y_n(t) \to y(t) \text{ weakly in } V_\sigma^*, \;\; n \to +\infty. \tag{2.26}$$

From (2.25) it follows that $\exists\{d_{n_k}, y_{n_k}\}_{k\geq 1} \subset \{d_n, y_n\}_{n\geq 1}$ such that

$$\text{for a.e. } t \in S \;\; \langle d_{n_k}(t), y_{n_k}(t) - y(t)\rangle_V \to 0, \;\; k \to +\infty.$$

Let $\Sigma_1 \subset S$ be a set of measure zero such that for $t \notin \Sigma_1$ d_{n_k}, $y_{n_k}(t)$, $y(t)$ are well-defined $\forall k \geq 1$ $d_{n_k}(t) \in A(y_{n_k}(t))$ $\forall k \geq 1$ and

$$\langle d_{n_k}(t), y_{n_k}(t) - y(t)\rangle_V \to 0, \;\; k \to +\infty.$$

In virtue of Conditions $(A1)$ and $(A3)$ we obtain

$$\forall t \notin \Sigma_1 \;\; \forall k \geq 1 \;\; \overline{\lim_{k\to+\infty}} \left(c_2\|y_{n_k}(t)\|_V^p - c_3 - c_1(1 + \|y_{n_k}(t)\|_V^{p-1})\|y(t)\|_V\right) \leq 0.$$

Thus $\forall t \notin \Sigma_1$

$$\overline{\lim_{k\to+\infty}} \|y_{n_k}(t)\|_V^p \leq c(c_1, c_2, c_3, p)(1 + \|y(t)\|_V^p).$$

Therefore, due to (2.26) we obtain that for a.e. $t \in S$ $y_{n_k}(t) \to y(t)$ weakly in V, $k \to +\infty$.

Lemma 2.4. *Let* $p > 1$, $A : V \to C_v(V^*)$ *satisfies Conditions* $(A1)$, $(A2)$ *and* $(A3)$. *Then* $\hat{A} : X \to C_v(X^*)$, $\hat{A}(y) = \{g \in X^* \,|\, g(t) \in A(y(t))$ *for a.e.* $t \in S\}$, $y \in X$, *is pseudomonotone on* W_σ.

Proof. Let $y_n \to y$ weakly in W_σ, $d_n \in \hat{A}(y_n)$ $\forall n \geq 1$ and $\overline{\lim}_{n\to+\infty} \langle d_n, y_n - y\rangle_X \leq 0$. We need to show that for all $\omega \in X$ there exists $g(\omega) \in \hat{A}(y)$ such that

$$\underline{\lim}_{n\to+\infty} \langle d_n, y_n - \omega\rangle_X \geq \langle g(\omega), y - \omega\rangle_X.$$

Suppose on the contrary that for some $\omega \in X$

$$\underline{\lim}_{n\to+\infty} \langle d_n, y_n - \omega\rangle_X < [\hat{A}(y), y - \omega]_-. \tag{2.27}$$

On the other hand in virtue of Lemmas 2.2 and 2.3 we have that $\exists\{d_{n_k}, y_{n_k}\}_{k\geq 1} \subset \{d_n, y_n\}_{n\geq 1}$ such that

$$\underline{\lim}_{n\to+\infty} \langle d_n, y_n - \omega\rangle_X = \lim_{k\to+\infty} \langle d_{n_k}, y_{n_k} - \omega\rangle_X \tag{2.28}$$

$$\text{for a.e. } t \in S \ \langle d_{n_k}(t), y_{n_k}(t) - y(t)\rangle_V \to 0,\ k \to +\infty, \tag{2.29}$$

$$\text{for a.e. } t \in S \ y_{n_k}(t) \to y(t) \text{ weakly in } V,\ k \to +\infty, \tag{2.30}$$

$$\int_S |\langle d_{n_k}(t), y_{n_k}(t) - y(t)\rangle_V| dt \to 0,\ k \to +\infty, \tag{2.31}$$

$$d_{n_k} \to d \text{ weakly in } X^*,\ k \to +\infty. \tag{2.32}$$

As V is separable Banach space then there exists a countable dense system of vectors $\{v_j\}_{j\geq 1} \subset V$.

We finish the proof into several steps.

Step 1. In virtue of the pseudomonotony of A, from (2.29), (2.30) it follows that

$$\text{for a.e. } t \in S\ \forall j \geq 1\ \underline{\lim}_{k\to+\infty} \langle d_{n_k}(t), y_{n_k}(t) - \omega_j\rangle_V \geq [A(y(t)), y(t) - \omega_j]_-, \tag{2.33}$$

where $\underline{\lim}_{k\to+\infty} \langle d_{n_k}(t), y_{n_k}(t) - \omega_j\rangle_V = \underline{\lim}_{k\to+\infty} \langle d_{n_k}(t), y(t) - \omega_j\rangle_V$.

Step 2. Due to Conditions $(A2)$ and $(A3)$ it follows that $\forall k, j \geq 1$, for a.e. $s \in S$

$$\langle d_{n_k}(s), y_{n_k}(s) - \omega_j\rangle_V \geq c_2\|y_{n_k}(s)\|_V^p - c_3 - c_1(1 + \|y_{n_k}(s)\|_V^{p-1})\|\omega_j\|_V.$$

Now using Young's inequality, we can obtain

$$c_1\|y_n(s)\|_V^{p-1}\|\omega_j\|_V \leq c_2\|y_n(s)\|_V^p + c_2^{\frac{-p}{q}}\|\omega_j\|_V^p c_1^p p^{-1} q^{\frac{-p}{q}}.$$

Letting

$$c_{4,j} = c_1\|\omega_j\|_V + c_2^{\frac{-p}{q}}\|\omega_j\|_V^p c_1^p p^{-1} q^{\frac{-p}{q}} + c_3 > 0,$$

we finally get

$$\forall k, j \geq 1, \text{ for a.e. } t \in S \ \langle d_{n_k}(s), y_{n_k}(s) - \omega_j \rangle_V \geq -c_{4,j}. \tag{2.34}$$

Step 3. From (2.31) and (2.32) we have that $\forall t_1, t_2 \in S$, $t_1 < t_2$,

$$\int_{t_1}^{t_2} \langle d_{n_k}(s), y_{n_k}(s) - \omega_j \rangle_V ds \to \int_{t_1}^{t_2} \langle d(s), y(s) - \omega_j \rangle_V ds. \tag{2.35}$$

Step 4. In virtue of (2.31), (2.29), (2.35) and Fatou's lemma $\forall j \geq 1$, $\forall t \in S$, $\forall h > 0 : t + h \in S$, we obtain

$$\begin{aligned}
&\int_t^{t+h} \langle d(s), y(s) - \omega_j \rangle_V ds \\
&= \varliminf_{k \to +\infty} \int_t^{t+h} [\langle d_{n_k}(s), y_{n_k}(s) - \omega_j \rangle_V + c_{4,j}] ds - c_{4,j} h \\
&\geq \int_t^{t+h} \varliminf_{k \to +\infty} [\langle d_{n_k}(s), y_{n_k}(s) - \omega_j \rangle_V + c_{4,j}] ds - c_{4,j} h \\
&= \int_t^{t+h} \varliminf_{k \to +\infty} \langle d_{n_k}(s), y(s) - \omega_j \rangle_V ds.
\end{aligned} \tag{2.36}$$

Because of $\forall \varphi \in L_1(S)$

$$\frac{1}{h} \int_0^h \varphi(s + \cdot) ds \to \varphi(\cdot) \text{ in } L_1(S), \ h \searrow 0,$$

we have:

$$\text{for a.e. } t \in S, \ \forall j \geq 1, \ \langle d(t), y(t) - \omega_j \rangle_V \geq \varliminf_{k \to +\infty} \langle d_{n_k}(t), y(t) - \omega_j \rangle_V \geq$$

$$\geq [A(y(t)), y(t) - \omega_j]_-.$$

Step 5. In virtue of $\{y(t) - \omega_j\}_{j \geq 1}$ is dense in V for a.e. $t \in S$ we finally obtain that $d(t) \in A(y(t))$ for a.e. $t \in S$, i.e. $d \in \hat{A}(y)$ and due to (2.31), (2.32), (2.28) we have

$$\lim_{n\to+\infty} \langle d_n, y_n - \omega\rangle_X = \langle d, y - \omega\rangle_X \geq [\hat{A}(y), y - \omega]_-,$$

that contradicts to (2.27).
Lemma 2.4 is proved.

2.3 Auxiliary Properties of Solutions for the First Order Evolution Inclusions with Uniformly Coercive Mappings

Let us consider the additional properties for the first order autonomous evolution inclusions.

2.3.1 *The Setting of the Problem*

For evolution triple $(V; H; V^*)$, multi-valued (in the general case) map $A : V \overset{\to}{\to} V^*$ and exciting force $f \in V^*$ we consider a problem of investigation of dynamics for all weak solutions defined for $t \geq 0$ of non-linear autonomous differential-operator inclusion

$$y'(t) + A(y(t)) \ni f, \tag{2.37}$$

as $t \to +\infty$ in the phase space H. Parameters of this problem satisfy the next properties:

(H_1) $p \geq 2$, $f \in V^*$;
(H_2) The embedding V into H is compact one;
(A_1) $\exists c > 0$: $\forall u \in V$, $\forall d \in A(u)$ $\quad \|d\|_{V^*} \leq c(1 + \|u\|_V^{p-1})$;
(A_2) $\exists \alpha, \beta > 0$: $\forall u \in V$, $\forall d \in A(u)$ $\quad \langle d, u\rangle_V \geq \alpha\|u\|_V^p - \beta$;
(A_3) $A : V \overset{\to}{\to} V^*$ is (generalized) pseudomonotone, i.e.

(a) For every $u \in V$ the set $A(u)$ is a nonempty, convex and weakly compact one in V^*;

(b) If $u_n \to u$ weakly in V, $d_n \in A(u_n)$ $\forall n \geq 1$ and $\overline{\lim}_{n\to+\infty} \langle d_n, u_n - u\rangle_V \leq 0$ then $\forall \omega \in V$ $\exists d(\omega) \in A(u)$:

$$\underline{\lim}_{n\to+\infty} \langle d_n, u_n - \omega\rangle_V \geq \langle d(\omega), u - \omega\rangle_V.$$

Here $\langle \cdot, \cdot\rangle_V : V^* \times V \to \mathbf{R}$ is a pairing in $V^* \times V$ coincident on $H \times V$ with the inner product $(\cdot, \cdot)$ in the Hilbert space H.

Remark 2.1. From properties (A_1)–(A_3) it follows that the map A is u.s.c. from every finite-dimensional subspace V into V^* equipped with the weak topology.

As a *weak solution* of evolution inclusion (2.37) on the interval $[\tau, T]$ we consider an element u of the space $L_p(\tau, T; V)$ such that for some $d \in L_q(\tau, T; V^*)$

$$d(t) \in A(y(t)) \quad \text{for almost each (a.e.) } t \in (\tau, T), \tag{2.38}$$

$$-\int_\tau^T (\xi'(t), u(t))dt + \int_\tau^T \langle d(t), \xi(t)\rangle_V dt = \int_\tau^T (f, \xi(t))dt \quad \forall \xi \in C_0^\infty([\tau, T]; V), \tag{2.39}$$

where $q > 1$: $\frac{1}{p} + \frac{1}{q} = 1$.

2.3.2 Preliminaries

For fixed $\tau < T$ let us consider

$$X_{\tau,T} = L_p(\tau, T; V), \quad X^*_{\tau,T} = L_q(\tau, T; V^*), \quad W_{\tau,T} = \{u \in X_{\tau,T} \mid u' \in X^*_{\tau,T}\},$$
$$\mathscr{A}_{\tau,T} : X_{\tau,T} \rightrightarrows X^*_{\tau,T}, \quad \mathscr{A}_{\tau,T}(y) = \{d \in X^*_{\tau,T} \mid d(t) \in A(y(t)) \text{ for a.e. } t \in (\tau, T)\},$$
$$f_{\tau,T} \in X^*_{\tau,T}, \quad f_{\tau,T}(t) = f \text{ for a.e. } t \in (\tau, T),$$

where u' is a derivative of an element $u \in X_{\tau,T}$ in the sense of the space of distributions $\mathscr{D}([\tau, T]; V^*)$ (see, for example, [42]). Note that the space $W_{\tau,T}$ is a reflexive Banach space with the graph norm of a derivative (see, for example [42]):

$$\|u\|_{W_{\tau,T}} = \|u\|_{X_{\tau,T}} + \|u'\|_{X^*_{\tau,T}}, \quad u \in W_{\tau,T}. \tag{2.40}$$

From Sect. 2.2, (A_1)–(A_3) it follows that $\mathscr{A}_{\tau,T} : X_{\tau,T} \rightrightarrows X^*_{\tau,T}$ satisfies properties:

(B_1) $\exists C_1 > 0$: $\|d\|_{X^*_{\tau,T}} \le C_1(1 + \|y\|^{p-1}_{X_{\tau,T}})$ $\forall y \in X_{\tau,T}$, $\forall d \in \mathscr{A}_{\tau,T}(y)$;
(B_2) $\exists C_2, C_3 > 0$: $\langle d, y\rangle_{X_{\tau,T}} \ge C_2\|y\|^p_{X_{\tau,T}} - C_3$ $\forall y \in X_{\tau,T}$, $\forall d \in \mathscr{A}_{\tau,T}(y)$;
(B_3) $\mathscr{A}_{\tau,T} : X_{\tau,T} \rightrightarrows X^*_{\tau,T}$ is (generalized) pseudomonotone on $W_{\tau,T}$ operator, i.e.

(a) For every $y \in X_{\tau,T}$ the set $\mathscr{A}_{\tau,T}(y)$ is a nonempty, convex and weakly compact one in $X^*_{\tau,T}$;

(b) $\mathscr{A}_{\tau,T}$ is u.s.c. from every finite dimensional subspace $X_{\tau,T}$ into $X^*_{\tau,T}$ endowed with the weak topology;

(c) If $y_n \to y$ weakly in $W_{\tau,T}$, $d_n \in \mathscr{A}_{\tau,T}(y_n)$ $\forall n \ge 1$, $d_n \to d$ weakly in $X^*_{\tau,T}$ and
$$\overline{\lim_{n\to+\infty}} \langle d_n, y_n - y\rangle_{X_{\tau,T}} \le 0$$
then $d \in \mathscr{A}_{\tau,T}(y)$ $\lim_{n\to+\infty} \langle d_n, y_n\rangle_{X_{\tau,T}} = \langle d, y\rangle_{X_{\tau,T}}$.

Here $\langle\cdot,\cdot\rangle_{X_{\tau,T}} : X^*_{\tau,T} \times X_{\tau,T} \to \mathbf{R}$ is a pairing in $X^*_{\tau,T} \times X_{\tau,T}$ coincident on $L_2(\tau, T; H) \times X_{\tau,T}$ with the inner product in $L_2(\tau, T; H)$, i.e.

$$\forall u \in L_2(\tau, T; H), \ \forall v \in X_{\tau,T} \quad \langle u, v \rangle_{X_{\tau,T}} = \int_\tau^T (u(t), v(t))dt.$$

Note also (see [16, Theorem IV.1.17, c. 177]) that the embedding $W_{\tau,T} \subset C([\tau, T]; H)$ is continuous and dense, moreover,

$$\forall u, v \in W_{\tau,T} \quad (u(T), v(T)) - (u(\tau), v(\tau)) = \int_\tau^T \Big[\langle u'(t), v(t)\rangle_V + \langle v'(t), u(t)\rangle_V\Big]dt. \tag{2.41}$$

From the definition of a derivative in the sense of $\mathscr{D}([\tau, T]; V^*)$ and equality (2.39) it directly follows such statement:

Lemma 2.5. *Each weak solution $u \in X_{\tau,T}$ of differential-operator inclusion (2.37) on the interval $[\tau, T]$ belongs to the space $W_{\tau,T}$. Moreover,*

$$u' + \mathscr{A}_{\tau,T}(u) \ni f_{\tau,T}. \tag{2.42}$$

Vice versa, if $u \in W_{\tau,T}$ satisfies (2.42) then u is a weak solution of (2.37) on $[\tau, T]$.

Properties (B_1)–(B_3), (H_1), [42] provide the existence of a weak solution of Cauchy problem (2.37) with initial data

$$y(\tau) = y_\tau \tag{2.43}$$

on the interval $[\tau, T]$ for an arbitrary $y_\tau \in H$. Therefore, the next result takes place:

Lemma 2.6. *$\forall \tau < T, y_\tau \in H$ Cauchy problem (2.37), (2.43) has a weak solution on the interval $[\tau, T]$. Moreover, each weak solution $u \in X_{\tau,T}$ of Cauchy problem (2.37), (2.43) on the interval $[\tau, T]$ belongs to $W_{\tau,T} \subset C([\tau, T]; H)$ and satisfies (2.42).*

Remark 2.2. Since $W_{\tau,T} \subset C([\tau, T]; H)$, for each weak solution of problem (2.37), in view of Lemma 2.5, initial data (2.43) has sense.

For fixed $\tau < T$ we denote

$$\mathscr{D}_{\tau,T}(u_\tau) = \{u(\cdot) \mid u \text{ is a weak solution of (2.37) on } [\tau, T], \ u(\tau) = u_\tau\}, \quad u_\tau \in H.$$

From Lemma 2.6 it follows that $\mathscr{D}_{\tau,T}(u_\tau) \neq \emptyset$ and $\mathscr{D}_{\tau,T}(u_\tau) \subset W_{\tau,T}$ $\forall \tau < T$, $u_\tau \in H$.

We complete this section checking that the translation and concatenation of weak solutions is a weak solution too.

Lemma 2.7. *If $\tau < T$, $u_\tau \in H$, $u(\cdot) \in \mathscr{D}_{\tau,T}(u_\tau)$, then $v(\cdot) = u(\cdot + s) \in \mathscr{D}_{\tau-s,T-s}(u_\tau)$ $\forall s$. If $\tau < t < T$, $u_\tau \in H$, $u(\cdot) \in \mathscr{D}_{\tau,t}(u_\tau)$ and $v(\cdot) \in \mathscr{D}_{t,T}(u(t))$, then*

$$z(s) = \begin{cases} u(s), \ s \in [\tau, t], \\ v(s), \ s \in [t, T] \end{cases}$$

belongs to $\mathscr{D}_{\tau,T}(u_\tau)$.

Proof. The proof follows from the definition of solution (2.39), Lemma 2.5 and from $z \in W_{\tau,T}$ as soon as $v \in W_{\tau,t}$, $u \in W_{t,T}$ and $v(t) = u(t)$. Proving the last statement we can use the definition of a derivative in the sense $\mathscr{D}([\tau, T]; V^*)$, (2.41) and [16, Chap. IV] on the density of $C^1([t_1, t_2]; V)$ in W_{t_1,t_2} for $t_1 < t_2$.

2.3.3 Supplementary Properties of Solutions

The proof of the existence of compact global and trajectory attractors for evolutional inclusions and, in particular, equations of type (2.37) as a rule is based on properties of a family of weak solutions (2.37), related to the absorbing of the generated m-semiflow of solutions and its asymptotic compactness (see, for example, works [21, 24, 36, 37] and references there). The next lemma on a priory estimates for solutions and Theorem 2.1 on dependence of solutions on initial data are “key players” when investigating the dynamics for solutions of problem (2.37) as $t \to +\infty$.

Lemma 2.8. *There exist* $c_4, c_5, c_6, c_7 > 0$ *such that for any finite interval of time* $[\tau, T]$ *every weak solution u of problem (2.37) on* $[\tau, T]$ *satisfies estimates:* $\forall t \geq s$, $t, s \in [\tau, T]$

$$\|u(t)\|_H^2 + c_4 \int_s^t \|u(\xi)\|_V^p d\xi \leq \|u(s)\|_H^2 + c_5 \left(1 + \|f\|_{V^*}^2\right)(t - s), \tag{2.44}$$

$$\|u(t)\|_H^2 \leq \|u(s)\|_H^2 e^{-c_6(t-s)} + c_7 \left(1 + \|f\|_{V^*}^2\right). \tag{2.45}$$

Proof. The proof naturally follows from conditions for the parameters of problem (2.37) and Gronwall lemma.

Theorem 2.1. *Let* $\tau < T$, $\{u_n\}_{n\geq 1}$ *be an arbitrary sequence of weak solutions of (2.37) on* $[\tau, T]$ *such that* $u_n(\tau) \to \eta$ *weakly in* H. *Then there exist* $\{u_{n_k}\}_{k\geq 1} \subset \{u_n\}_{n\geq 1}$ *and* $u(\cdot) \in \mathscr{D}_{\tau,T}(\eta)$ *such that*

$$\forall \varepsilon \in (0, T - \tau) \qquad \max_{t \in [\tau+\varepsilon, T]} \|u_{n_k}(t) - u(t)\|_H \to 0, \quad k \to +\infty. \tag{2.46}$$

Proof. Suppose that conditions of Theorem 2.1 are satisfied. Then in view of Lemma 2.5 for any $n \geq 1$ $u_n(\cdot) \in W_{\tau,T} \subset C([\tau, T]; H)$. Moreover, from Lemma 2.8, property (A_2) and relation (2.42) we have that

$$\forall n \geq 1 \quad \exists d_n \in \mathscr{A}_{\tau,T}(u_n): \quad u_n'(t) + d_n(t) = f \text{ for a.e. } t \in (\tau, T), \tag{2.47}$$

$$\exists C > 0: \quad \forall n \geq 1 \quad \|u_n\|_{X_{\tau,T}} + \|u_n'\|_{X^*_{\tau,T}} + \|u_n\|_{C([\tau,T];H)} + \|d_n\|_{X^*_{\tau,T}} \leq C. \tag{2.48}$$

Hence, due to the continuous embedding $W_{\tau,T} \subset C([\tau,T];H)$ [16, Chap. IV], properties (H_2) (B_1), the compactness of the embedding $W_{\tau,T} \subset L_2(\tau,T;H)$ (see [22, Chap. 1]), and the reflexivity of the space $W_{\tau,T}$ with the graph norm of a derivative (2.40), we obtain that up to a subsequence $\{u_{n_k}, d_{n_k}\}_{k\geq 1} \subset \{u_n, d_n\}_{n\geq 1}$ for some $u \in W_{\tau,T}$, $d \in X^*_{\tau,T}$ the next convergence take place:

$$\begin{aligned} u_{n_k} &\to u \text{weakly in } X_{\tau,T},\\ u'_{n_k} &\to u' \text{weakly in } X^*_{\tau,T},\\ d_{n_k} &\to d \text{weakly in } X^*_{\tau,T},\\ u_{n_k} &\to u \text{weakly in } C([\tau,T];H),\\ u_{n_k} &\to u \text{in } L_2(\tau,T;H),\\ u_{n_k}(t) &\to u(t) \text{in } H \text{ for a.e. } t \in (\tau,T), \quad k \to +\infty. \end{aligned} \tag{2.49}$$

Let us complete the proof of this theorem in a few “steps”.

Step 1. Prove that

$$\forall t \in (\tau,T] \quad u_{n_k}(t) \to u(t) \text{ in } H,\ k \to +\infty. \tag{2.50}$$

From Lemma 2.8 it follows that $\forall k \geq 1\ \forall t \geq s,\ t,s \in [\tau,T]$

$$\|u_{n_k}(t)\|_H^2 - c_5(1+\|f\|_H^2)t \leq \|u_{n_k}(s)\|_H^2 - c_5(1+\|f\|_H^2)s. \tag{2.51}$$

Moreover, from (2.49) we have that for a.e. $s \in (\tau,T)$ for a.e. $t \in (s,T)$

$$\|u(t)\|_H^2 - c_5(1+\|f\|_H^2)t \leq \|u(s)\|_H^2 - c_5(1+\|f\|_H^2)s.$$

Since $u \in W_{\tau,T} \subset C([\tau,T];H)$, then $\forall t \geq s\ t,s \in [\tau,T]$

$$\|u(t)\|_H^2 - c_5(1+\|f\|_H^2)t \leq \|u(s)\|_H^2 - c_5(1+\|f\|_H^2)s. \tag{2.52}$$

Therefore, functions

$$J_k(t) = \|u_{n_k}(t)\|_H^2 - c_5(1+\|f\|_H^2)t, \tag{2.53}$$

$$J(t) = \|u(t)\|_H^2 - c_5(1+\|f\|_H^2)t, \tag{2.54}$$

are continuous and monotone nonincreasing one on $[\tau,T]$.
Moreover, since $u_{n_k}(t) \to u(t)$ in H for a.e. $t \in (\tau,T)$, then

$$J_k(t) \to J(t),\ k \to +\infty \text{ for a.e. } t \in (\tau,T). \tag{2.55}$$

Show that

$$\overline{\lim_{k\to+\infty}} J_k(t) \le J(t) \quad \forall t \in (\tau, T]. \tag{2.56}$$

From (2.55) it follows that

$$\forall t \in (\tau, T], \forall \varepsilon > 0\, \exists \bar{t} \in (\tau, t) : \ |J(\bar{t}) - J(t)| < \varepsilon \text{ and } \lim_{k\to+\infty} J_k(\bar{t}) = J(\bar{t}).$$

Hence, $\forall k \ge 1$

$$J_k(t) - J(t) \le J_k(\bar{t}) - J(t) \le |J_k(\bar{t}) - J(\bar{t})| + |J(\bar{t}) - J(t)| < \varepsilon + |J_k(\bar{t}) - J(\bar{t})|.$$

Therefore,

$$\forall t \in (\tau, T],\ \forall \varepsilon > 0\ \overline{\lim_{k\to+\infty}} J_k(t) \le J(t) + \varepsilon.$$

Hence (2.56) and, in particular, the inequality

$$\overline{\lim_{k\to+\infty}} \|u_{n_k}(t)\|_H^2 \le \|u(t)\|_H^2 \quad \forall t \in (\tau, T]$$

are true. From weak convergence $u_{n_k}(t)$ to $u(t)$ in H as $k \to +\infty$ $\forall t \in [\tau, T]$, inequality (2.56) and [16, Chap. I] we obtain (2.50).

Step 2. Show that

$$u' = f_{\tau,T} - d. \tag{2.57}$$

In view of Lemma 2.5 for any $k \ge 1$, $\xi \in C_0^\infty([\tau, T]; V)$ we have

$$-\langle \xi', u_{n_k} \rangle_{X_{\tau,T}} + \langle d_{n_k}, \xi \rangle_{X_{\tau,T}} = \langle f_{\tau,T}, \xi \rangle. \tag{2.58}$$

Passing to the limit as $k \to +\infty$ in the last relation we obtain

$$\forall \xi \in C_0^\infty([\tau, T]; V)\ -\langle \xi', u \rangle_{X_{\tau,T}} + \langle d, \xi \rangle_{X_{\tau,T}} = \langle f_{\tau,T}, \xi \rangle.$$

Therefore, using properties of the Bochner integral, we obtain $\forall \varphi \in C_0^\infty([\tau, T])$ $\forall h \in V$

$$-\left(\int_\tau^T u(s)\varphi'(s)ds, h \right) = -\int_\tau^T (h, u(s))_H \varphi'(s)ds$$

$$= \int_\tau^T \langle f - d(s), h \rangle_V \varphi(s)ds = \left\langle \int_\tau^T [f_{\tau,T}(s) - d(s)]\varphi(s)ds, h \right\rangle_V .$$

From the definition of a derivative of an element $u \in X_{\tau,T}$ in the sense of $\mathscr{D}^*([\tau, T]; V^*)$ it directly follows relation (2.57).

Step 3. Fix an arbitrary $\varepsilon \in (0, T - \tau)$ and show that

$$d(t) \in A(u(t)) \text{ for a.e. } t \in (\tau + \varepsilon, T), \tag{2.59}$$

using the pseudomonotony of $\mathscr{A}_{\tau+\varepsilon,T}$ on $W_{\tau+\varepsilon,T}$.
Consider restrictions $u_{n_k}(\cdot)$, $d_{n_k}(\cdot)$, $u(\cdot)$, $d(\cdot)$ to the interval $[\tau+\varepsilon, T]$. To simplify the consideration we denote them by the same symbols: $u_{n_k}(\cdot)$, $d_{n_k}(\cdot)$, $u(\cdot)$ and $d(\cdot)$ respectively. From convergence (2.49), (2.50) we have that

$$\begin{aligned}
&u_{n_k} \to u \text{ weakly in } W_{\tau+\varepsilon,T},\\
&d_{n_k} \to d \text{ weakly in } X^*_{\tau+\varepsilon,T},\\
&\forall t \in [\tau + \varepsilon, T] \quad u_{n_k}(t) \to u(t) \text{ in } H,\ k \to +\infty.
\end{aligned} \tag{2.60}$$

Show that

$$\lim_{k\to+\infty} \langle d_{n_k}, u_{n_k} - u\rangle_{X_{\tau+\varepsilon,T}} = 0. \tag{2.61}$$

Indeed,

$$\begin{aligned}
&\forall k \geq 1 \int_{\tau+\varepsilon}^{T} \langle d_{n_k}(s), u_{n_k}(s) - u(s)\rangle_V \, ds\\
&= \int_{\tau+\varepsilon}^{T} \big(f, u_{n_k}(s) - u(s)\big)\, ds - \int_{\tau+\varepsilon}^{T} \big\langle u'_{n_k}(s), u_{n_k}(s) - u(s)\big\rangle_V \, ds.
\end{aligned} \tag{2.62}$$

From (2.60) it follows that

$$\int_{\tau+\varepsilon}^{T} \big(f, u_{n_k}(s) - u(s)\big)\, ds \to 0,\ k \to +\infty. \tag{2.63}$$

From (2.41) and (2.60) we obtain that

$$\begin{aligned}
&\int_{\tau+\varepsilon}^{T} \langle u'_{n_k}(s), u(s) - u_{n_k}(s)\rangle_V \, ds\\
&= \int_{\tau+\varepsilon}^{T} \langle u'_{n_k}(s), u(s)\rangle_V - \frac{1}{2}\left(\|u_{n_k}(T)\|_H^2 - \|u_{n_k}(\tau + \varepsilon)\|_H^2\right)\\
&\to \int_{\tau+\varepsilon}^{T} \langle u'(s), u(s)\rangle_V - \frac{1}{2}\left(\|u(\tau)\|_H^2 - \|u(\tau + \varepsilon)\|_H^2\right) = 0,\ k \to +\infty.
\end{aligned} \tag{2.64}$$

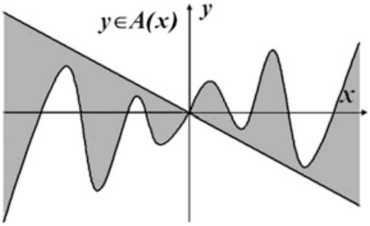

Fig. 2.5 The weakly "+"-coercive, but not weakly "−"-coercive multivalued map

Pass to the limit as $k \to +\infty$ in (2.62). From (2.63) and (2.64) we obtain (2.61). So, due to (2.47), (2.60), (2.61) and in view of the pseudomonotony of $\mathscr{A}_{\tau+\varepsilon,T}$ on $W_{\tau+\varepsilon,T}$ we obtain (2.59).

Step 4. From the arbitrariness of $\varepsilon \in (0, T - \tau)$, convergence (2.49), relation (2.59) and the definition of $\mathscr{A}_{\tau,T}$ it follows that $u(\cdot) \in \mathscr{D}_{\tau,T}(\eta)$.

Step 5. Let us prove (2.46). By contradiction suppose the existence of $\varepsilon > 0$, $L > 0$ and subsequence $\{u_{k_j}\}_{j\geq 1} \subset \{u_{n_k}\}_{k\geq 1}$ such that

$$\forall j \geq 1 \qquad \max_{t\in[\tau+\varepsilon,T]} \|u_{k_j}(t) - u(t)\|_H = \|u_{k_j}(t_j) - u(t_j)\|_H \geq L.$$

Without loss of generality we suggest that $t_j \to t_0 \in [\tau+\varepsilon, T]$, $j \to +\infty$. Therefore, by virtue of the continuity of $u : [\tau, T] \to H$, we have

$$\varliminf_{j\to+\infty} \|u_{k_j}(t_j) - u(t_0)\|_H \geq L. \tag{2.65}$$

On the other hand we prove that

$$u_{k_j}(t_j) \to u(t_0) \text{ in } H,\ \ j \to +\infty. \tag{2.66}$$

Step 5.1. Firstly let us show that

$$u_{k_j}(t_j) \to u(t_0) \text{ weakly in } H, \quad j \to +\infty. \tag{2.67}$$

For a fixed $h \in V$ from (2.49) it follows that the sequence of real functions $(u_{n_k}(\cdot), h) : [\tau, T] \to \mathbf{R}$ is uniformly bounded and equicontinuous. Taking into account inequality (2.48) and the density of embedding $V \subset H$ we obtain that $u_{n_k}(t) \to u(t)$ weakly in H uniformly on $[\tau, T]$, $k \to +\infty$. So, we obtain (2.67) (Fig. 2.5).

Step 5.2. Let us prove that

$$\varlimsup_{j\to+\infty} \|u_{k_j}(t_j)\|_H \leq \|u(t_0)\|_H. \tag{2.68}$$

We consider continuous nonincreasing functions J_{k_j}, J, $j \geq 1$, defined in (2.53), (2.54). Let us fix an arbitrary $\varepsilon_1 > 0$. From (2.55) and from the continuity of J it follows that

$$\exists \bar{t} \in (\tau, t_0) : \quad \lim_{j \to +\infty} J_{k_j}(\bar{t}) = J(\bar{t}) \text{ and } |J(\bar{t}) - J(t_0)| < \varepsilon_1.$$

Then for rather large $j \geq 1$

$$J_{k_j}(t_j) - J(t_0) \leq |J_{k_j}(\bar{t}) - J(\bar{t})| + |J(\bar{t}) - J(t_0)| \leq |J_{k_j}(\bar{t}) - J(\bar{t})| + \varepsilon_1.$$

Therefore, $\overline{\lim}_{j \to +\infty} J_{k_j}(t_j) \leq J(t_0) + \varepsilon_1$. From the arbitrariness of $\varepsilon_1 > 0$ and from $t_j \to t_0$, $j \to +\infty$, we obtain (2.68).

Step 5.3. Convergence (2.66) directly follows from (2.67), (2.68) and [16, Chap. I].

Step 5.4. To finish the proof of the theorem we remark that (2.66) contradicts (2.65). Therefore, (2.46) is true.

Corollary 2.1. *Let $\tau < T$, $\{u_n\}_{n \geq 1}$ be an arbitrary sequence of weak solutions of (2.37) on $[\tau, T]$ such that $u_n(\tau) \to \eta$ in H, $n \to +\infty$. Then there exists $u(\cdot) \in \mathscr{D}_{\tau,T}(\eta)$ and $\{u_{n_k}\}_{k \geq 1} \subset \{u_n\}_{n \geq 1}$ such that $u_{n_k} \to u$ in $C([\tau, T]; H)$, $k \to +\infty$.*

Proof. The proof is similar to the proof of Theorem 2.1. The main difference is in the checking of the inequality $\overline{\lim}_{j \to +\infty} J_{k_j}(t_j) \leq J(t_0)$, when $t_0 = \tau$, $t_j \to t_0$, $j \to +\infty$, $\{t_j\}_{j \geq 1} \subset [\tau, T]$ (see Step 5.2 from the proof of Theorem 2.1). In this case $\forall j \geq 1$ $J_{k_j}(t_j) - J(\tau) \leq J_{k_j}(\tau) - J(\tau)$. Since $u_n(\tau) \to u(\tau)$ in H, $n \to +\infty$, then $J_{k_j}(\tau) \to J(\tau)$, $j \to +\infty$. Therefore, $\overline{\lim}_{j \to +\infty} J_{k_j}(t_j) \leq J(t_0)$.

2.4 Asymptotic Behavior of the First Order Evolution Inclusions

We note that problem (2.37) arises in many important models for distributed parameter control problems and that large class of identification problems enter our formulation. Let us indicate a problem which is one of motivations for the study of the autonomous evolution inclusion (2.37) [26]. In a subset Ω of $\mathbf{R}^3$, we consider the nonstationary heat conduction equation

$$\frac{\partial y}{\partial t} - \triangle y = f \text{ in } \Omega \times (0, +\infty)$$

with initial conditions and suitable boundary ones. Here $y = y(x,t)$ represents the temperature at the point $x \in \Omega$ and time $t > 0$. It is supposed that $f = \bar{f} + \bar{\bar{f}}$, where $\bar{\bar{f}}$ is given and $\bar{f}$ is a known function of the temperature of the form

$$-\bar{f}(x,t) \in \partial j(x, y(x,t)) \text{ a.e. } (x,t) \in \Omega \times (0, +\infty).$$

Here $\partial j(x,\xi)$ denotes generalized gradient of Clarke with respect to the last variable of a function $j : \Omega \times \mathbf{R} \to \mathbf{R}$ which is assumed to be locally Lipschitz in ξ. The multivalued function $\partial j(x,\cdot) : \mathbf{R} \to 2^{\mathbf{R}}$ is generally nonmonotone and it includes the vertical jumps. In a physicist's language it means that the law is characterized by the generalized gradient of a nonsmooth potential j.

The variational formulation of the above problem leads to the inclusion (6.5) with, for example, $H = L_2(\Omega)$, $V = H^1(\Omega)$, $A = -\triangle + \partial j$ and it is met, for example, in nonmonotone nonconvex interior semipermeability problems.

We remark that monotone semipermeability problems, leading to variation inequalities, have been studied in [15] under the assumption that $j(x,t)$ is a proper, lower semicontinuous, convex function which means that $\partial j(x,\cdot)$ is maximal monotone in $\mathbf{R}^2$.

Following the upper presented results under similar conditions to [15, 26, 30] we can state not only the existence of solutions for autonomous evolution objects but also investigate the dynamic of all weak solutions as $t \to +\infty$. We can also consider other examples from [15, 26, 30].

2.4.1 *Existence of the Global Attractor*

First we consider constructions presented in [24]. Denote the set of all nonempty (nonempty bounded) subsets of H by $P(H)$ ($\mathscr{B}(H)$). We recall that the multivalued map $G : \mathbf{R} \times H \to P(H)$ is said to be a *m-semiflow* if:

(a) $G(0,\cdot) = \mathrm{Id}$ (the identity map),

(b) $G(t+s,x) \subset G(t,G(s,x))\ \forall x \in H, t,s \in \mathbf{R}_+$;
m-semiflow *is a strict* one if $G(t+s,x) = G(t,G(s,x))\ \forall x \in H, t,s \in \mathbf{R}_+$.

From Lemmas 2.7 and 2.8 it follows that any weak solution can be extended to a global one defined on $[0,+\infty)$. For an arbitrary $y_0 \in H$ let $\mathscr{D}(y_0)$ be the set of all weak solutions (defined on $[0,+\infty)$) of problem (6.5) with initial data $y(0) = y_0$.

We define the m-semiflow G as $G(t,y_0) = \{y(t) \mid y(\cdot) \in \mathscr{D}(y_0)\}$.

Lemma 2.9. *G is the strict m-semiflow.*

Proof. Let $y \in G(t+s,y_0)$. Then $y = u(t+s)$, where $u(\cdot) \in \mathscr{D}(y_0)$. From Lemma 2.7 it follows that $v(\cdot) = u(s+\cdot) \in \mathscr{D}(u(s))$. Hence $y = v(t) \in G(t,u(s)) \subset G(t,G(s,y_0))$.

Vice versa, if $y \in G(t,G(s,y_0))$, then $\exists u(\cdot) \in \mathscr{D}(y_0)\ v(\cdot) \in \mathscr{D}(u(s))$: $y = v(t)$. Define the map

$$z(\xi) = \begin{cases} u(\xi), & \xi \in [0,s], \\ v(\xi - s), & \xi \in [s,t+s]. \end{cases}$$

From Lemma 2.7 it follows that $z(\cdot) \in \mathscr{D}(y_0)$. Hence $y = z(t+s) \in G(t+s,y_0)$.

We recall that the set $\mathscr{A}$ is said to be *a global attractor* G, if:

1. $\mathscr{A}$ is negatively semiinvariant (i.e. $\mathscr{A} \subset G(t, \mathscr{A})\ \forall t \geq 0$);
2. $\mathscr{A}$ is attracting, that is,

$$\operatorname{dist}(G(t, B), \mathscr{A}) \to 0, \quad t \to +\infty \quad \forall B \in \mathscr{B}(H), \tag{2.69}$$

   where $\operatorname{dist}(C, D) = \sup_{c \in C} \inf_{d \in D} \|c - d\|_H$ is the Hausdorff semidistance;
3. For any closed set $Y \subset H$ satisfying (2.69), we have $\mathscr{A} \subset Y$ (minimality). The global attractor is said to be *invariant* if $\mathscr{A} = G(t, \mathscr{A})\ \forall t \geq 0$. We prove the existence of the global attractor.

Theorem 2.2. *The m-semiflow G has the invariant compact in the phase space H global attractor $\mathscr{A}$.*

Proof. From Lemma 2.8 it follows that

$$\exists R, \tilde{\alpha} > 0: \quad \forall y_0 \in H,\ y(\cdot) \in \mathscr{D}(y_0),\ t \geq 0 \quad \|y(t)\|_H^2 \leq \|y_0\|_H^2 e^{-\tilde{\alpha} t} + R. \tag{2.70}$$

Therefore the ball $B_0 = \{u \in H \mid \|u\|_H \leq \sqrt{R+1}\}$ is the absorbing set, i.e. $\forall B \in \mathscr{B}(H)\ \exists T(B) > 0$: $\forall t \geq T(B)\ G(t, B) \subset B_0$. In particular, from (2.70) it follows that the set $\cup_{t \geq 0} G(t, B)$ is bounded one in $H\ \forall B \in \mathscr{B}(H)$.

Note also that from Theorem 2.1 it follows that the map $G(t, \cdot) : H \to \mathscr{B}(H)$ takes compact values and it is compact for $t > 0$ in that sense that it maps bounded sets into precompact one.

Show that the map $u_0 \to G(t, u_0)$ is upper semicontinuous [2, Definition 1.4.1, p. 38]. In order to do that it is sufficient to show [3, p. 45], that $\forall u_0 \in H$, $\forall \varepsilon > 0$ $\exists \delta(u_0, \varepsilon) > 0$: $\forall u \in B_\delta(u_0)\ G(t, u) \subset B_\varepsilon(G(t, u_0)) = \{z \in H \mid \operatorname{dist}(z, G(t, u_0)) < \varepsilon\}$. If it is not true then there exist $u_0 \in H$, $\varepsilon > 0$, $\{\delta_n\}_{n \geq 1} \subset (0, +\infty)$, $\{u_n\}_{n \geq 1} \subset H$ such that $\forall n \geq 1\ u_n \in B_{\delta_n}(u_0)$, $G(t, u_n) \not\subset B_\varepsilon(G(t, u_0))$ and $\delta_n \to 0$, $n \to +\infty$. Then $\forall n \geq 1\ \exists v_n(\cdot) \in \mathscr{D}(u_n)$: $v_n(t) \notin B_\varepsilon(G(t, u_0))$. Since $u_n \to u_0$ in H, $n \to +\infty$, then from Theorem 2.1 it follows that $v_n(t) \to v(t) \in G(t, u_0)$ in H, $n \to +\infty$, for some $v(\cdot) \in \mathscr{D}(u_0)$. We obtain contradiction with $\forall n \geq 1$ $\|v_n(t) - v(t)\|_H \geq \varepsilon$.

Thus the existence of the global attractor with required properties directly follows from results from Chap. 1.

2.4.2 Existence of the Trajectory Attractor

Let us consider the family $\mathscr{K}_+ = \cup_{y_0 \in H} \mathscr{D}(y_0)$ of all weak solutions of inclusion (6.5) defined on the semi-infinite interval $[0, +\infty)$. Note that $\mathscr{K}_+$ is *translation invariant* one, i.e. $\forall u(\cdot) \in \mathscr{K}_+$, $\forall h \geq 0\ u_h(\cdot) \in \mathscr{K}_+$, where $u_h(s) = u(h + s)$,

$s \geq 0$. We set the *translation semigroup* $\{T(h)\}_{h\geq 0}$, $T(h)u(\cdot) = u_h(\cdot)$, $h \geq 0$, $u \in \mathcal{K}_+$ on $\mathcal{K}_+$.

We shall construct the attractor of the translation semigroup $\{T(h)\}_{h\geq 0}$ acting on $\mathcal{K}_+$. On $\mathcal{K}_+$ we consider a topology induced from the Fréchet space $C^{loc}(\mathbf{R}_+; H)$. Note that

$$f_n(\cdot) \to f(\cdot) \text{ in } C^{loc}(\mathbf{R}_+; H) \iff \forall M > 0\ \Pi_M f_n(\cdot) \to \Pi_M f(\cdot) \text{ in } C([0, M]; H),$$

where Π_M is the restriction operator to the interval $[0, M]$ [37, p. 18]. We denote the restriction operator to the semi-infinite interval $[0, +\infty)$ by Π_+.

We recall that the a $\mathcal{P} \subset C^{loc}(\mathbf{R}_+; H) \cap L_\infty(\mathbf{R}_+; H)$ is said to be *attracting* for the trajectory space $\mathcal{K}_+$ of inclusion (6.5) in the topology of $C^{loc}(\mathbf{R}_+; H)$ if for any bounded in $L_\infty(\mathbf{R}_+; H)$ set $\mathcal{B} \subset \mathcal{K}_+$ and any number $M \geq 0$ the following relation holds:

$$\text{dist}_{C([0,M];H)}(\Pi_M T(t)\mathcal{B}, \Pi_M \mathcal{P}) \to 0, \quad t \to +\infty. \tag{2.71}$$

A set $\mathcal{U} \subset \mathcal{K}_+$ is said to be *trajectory attractor* in the trajectory space $\mathcal{K}_+$ with respect to the topology of $C^{loc}(\mathbf{R}_+; H)$ (see, for example, [37, Definition 1.2, p. 197]) if

(i) $\mathcal{U}$ is a compact set in $C^{loc}(\mathbf{R}_+; H)$ and bounded in $L_\infty(\mathbf{R}_+; H)$;
(ii) $\mathcal{U}$ is strictly invariant with respect to $\{T(h)\}_{h\geq 0}$, i.e. $T(h)\mathcal{U} = \mathcal{U}\ \forall h \geq 0$;
(iii) $\mathcal{U}$ is an attracting set in the trajectory space $\mathcal{K}_+$ in the topology $C^{loc}(\mathbf{R}_+; H)$.

Let us consider inclusions (6.5) on the entire time axis. Similarly to the space $C^{loc}(\mathbf{R}_+; H)$ the space $C^{loc}(\mathbf{R}; H)$ is equipped with the topology of local uniform convergence on each interval $[-M, M] \subset \mathbf{R}$ (see, for example, [37, p. 198]). A function $u \in C^{loc}(\mathbf{R}; H) \cap L_\infty(\mathbf{R}; H)$ is called a *complete trajectory* of inclusion (6.5) if $\forall h \in \mathbf{R}\ \Pi_+ u_h(\cdot) \in \mathcal{K}_+$ [37, p. 198]. Let $\mathcal{K}$ be a family all complete trajectories of inclusion (6.5). Note that

$$\forall h \in \mathbf{R},\ \forall u(\cdot) \in \mathcal{K} \quad u_h(\cdot) \in \mathcal{K}. \tag{2.72}$$

Lemma 2.10. *The set $\mathcal{K}$ is nonempty, compact in $C^{loc}(\mathbf{R}; H)$ and bounded in $L_\infty(\mathbf{R}; H)$. Moreover,*

$$\forall y(\cdot) \in \mathcal{K},\ \forall t \in \mathbf{R} \quad y(t) \in \mathcal{A}, \tag{2.73}$$

where $\mathcal{A}$ is the global attractor from Theorem 2.2.

Proof.

Step 1. Let us show that $\mathcal{K} \neq \emptyset$. Note that in view of [22, Theorem 3.1.1, p. 329] and conditions (A_1)–(A_3), (H_1), it follows that $\exists v \in V$: $A(v) = f$. We set $u(t) = v\ \forall t \in \mathbf{R}$. Then, $u \in \mathcal{K} \neq \emptyset$.

Step 2. Let us prove (2.73). For any $y \in \mathscr{K}$ $\exists d > 0$: $\|y(t)\|_H \le d$ $\forall t \in \mathbf{R}$. We set $B = \cup_{t\in\mathbf{R}}\{y(t)\} \in \mathscr{B}(H)$. Note that $\forall \tau \in \mathbf{R}$, $\forall t \in \mathbf{R}_+$ $y(\tau) = y_{\tau-t}(t) \in G(t, y_{\tau-t}(0)) \subset G(t, B)$. From Theorem 2.2 and from (2.69) it follows that $\forall \varepsilon > 0$ $\exists T > 0$: $\forall \tau \in \mathbf{R}$ $\operatorname{dist}(y(\tau), \mathscr{A}) \le \operatorname{dist}(G(T, B), \mathscr{A}) < \varepsilon$. Hence taking into account the compactness of $\mathscr{A}$ in H, for any $u(\cdot) \in \mathscr{K}$, $\tau \in \mathbf{R}$ it follows that $u(\tau) \in \mathscr{A}$.

Step 3. The boundedness of $\mathscr{K}$ in $L_\infty(\mathbf{R}_+; H)$ it follows from (2.73) and from the boundedness of $\mathscr{A}$ in H.

Step 4. Let us check the compactness of $\mathscr{K}$ in $C^{loc}(\mathbf{R}; H)$. In order to do that it is sufficient to check the precompactness and completeness.

Step 4.1. Let us check the precompactness of $\mathscr{K}$ in $C^{loc}(\mathbf{R}; H)$. If it is not true then in view of (2.72), $\exists M > 0$: $\Pi_M\mathscr{K}$ is not precompact in $C([0, M]; H)$. Hence there exists a sequence $\{v_n\}_{n\ge1} \subset \Pi_M\mathscr{K}$, that has not a convergent in $C([0, M]; H)$ subsequence. On the other hand $v_n = \Pi_M u_n$, where $u_n \in \mathscr{K}$, $v_n(0) = u_n(0) \in \mathscr{A}$, $n \ge 1$. Since $\mathscr{A}$ is compact in H (see Theorem 2.2), then in view of Corollary 2.1, $\exists\{v_{n_k}\}_{k\ge1} \subset \{v_n\}_{n\ge1}$, $\exists \eta \in H$, $\exists v(\cdot) \in \mathscr{D}_{0,M}(\eta)$: $v_{n_k}(0) \to \eta$ in H, $v_{n_k} \to v$ in $C([0, T]; H)$, $k \to +\infty$. We obtained contradiction.

Step 4.2. Let us check the completeness of $\mathscr{K}$ in $C^{loc}(\mathbf{R}; H)$. Let $\{v_n\}_{n\ge1} \subset \mathscr{K}$, $v \in C^{loc}(\mathbf{R}; H)$: $v_n \to v$ in $C^{loc}(\mathbf{R}; H)$, $n \to +\infty$. From the boundedness of $\mathscr{K}$ in $L_\infty(\mathbf{R}; H)$ it follows that $v \in L_\infty(\mathbf{R}; H)$. From Corollary 2.1 we have that $\forall M > 0$ the restriction $v(\cdot)$ to the interval $[-M, M]$ belongs to $\mathscr{D}_{-M,M}(v(-T))$. Therefore, $v(\cdot)$ is complete trajectory of inclusion (6.5). Thus, $v \in \mathscr{K}$.

Lemma 2.11. *Let $\mathscr{A}$ be a global attractor from Theorem 2.2. Then*

$$\forall y_0 \in \mathscr{A} \quad \exists y(\cdot) \in \mathscr{K}: \quad y(0) = y_0. \tag{2.74}$$

Proof. Let $y_0 \in \mathscr{A}$, $u(\cdot) \in \mathscr{D}(y_0)$. From (6.17), (2.69) we have that $\forall t \in \mathbf{R}_+$ $y(t) \in \mathscr{A}$. From Theorem 2.2 it follows that $G(1, \mathscr{A}) = \mathscr{A}$. Therefore,

$$\forall \eta \in \mathscr{A} \quad \exists \xi \in \mathscr{A},\ \exists \varphi_\eta(\cdot) \in \mathscr{D}_{0,1}(\xi): \quad \varphi_\eta(1) = \eta.$$

For any $t \in \mathbf{R}$ we set

$$y(t) = \begin{cases} u(t), & t \in \mathbf{R}_+, \\ \varphi_{y(-k+1)}(t+k), & t \in [-k, -k+1),\ k \in \mathbf{N}. \end{cases}$$

Note that $y \in C^{loc}(\mathbf{R}; H)$, $y(t) \in \mathscr{A}$ $\forall t \in \mathbf{R}$ (consequently $y \in L_\infty(\mathbf{R}; H)$) and in view of Lemma 2.7, $y \in \mathscr{K}$. At that $y(0) = y_0$.

Theorem 2.3. *Let $\mathscr{A}$ be a global attractor from Theorem 2.2. Then there exists the trajectory attractor $\mathscr{P} \subset \mathscr{K}_+$ in the space $\mathscr{K}_+$. At that the next formula holds:*

$$\mathscr{P} = \Pi_+\mathscr{K} = \Pi_+\{y \in \mathscr{K} \mid y(t) \in \mathscr{A}\ \forall t \in \mathbf{R}\}, \tag{2.75}$$

Proof. From Lemma 2.10 and the continuity of operator $\Pi_+ : C^{loc}(\mathbf{R}; H) \to C^{loc}(\mathbf{R}_+; H)$ it follows that the set $\Pi_+\mathscr{K}$ is nonempty, compact in $C^{loc}(\mathbf{R}_+; H)$ and bounded one in $L_\infty(\mathbf{R}_+; H)$. Moreover, the second equality in (2.75) holds. The strict invariance of $\Pi_+\mathscr{K}$ follows from the autonomy of inclusion (6.5).

Let us prove that $\Pi_+\mathscr{K}$ is the attracting set for the trajectory space $\mathscr{K}_+$ in the topology $C^{loc}(\mathbf{R}_+; H)$. Let $B \subset \mathscr{K}_+$ be a bounded set in $L_\infty(\mathbf{R}_+; H)$, $M \geq 0$. Let us check (2.71). If it is not true then there exist sequences $t_n \to +\infty$, $v_n(\cdot) \in B$ such that

$$\forall n \geq 1 \quad \mathrm{dist}_{C([0,T];H)}(\Pi_M v_n(t_n + \cdot), \Pi_M \mathscr{K}) \geq \varepsilon. \tag{2.76}$$

On the other hand, from the boundedness B in $L_\infty(\mathbf{R}_+; H)$ it follows that $\exists R > 0$: $\forall v(\cdot) \in B$, $\forall t \in \mathbf{R}_+$ $\|v(t)\|_H \leq R$. Hence, $\exists N \geq 1$: $\forall n \geq N$ $v_n(t_n) \in G(t_n, v_n(0)) \subset G(1, G(t_n - 1, v_n(0))) \subset G(1, \overline{B_R})$, where $\overline{B_R} = \{u \in H \mid \|u\|_H \leq R\}$. Therefore, taking into account (2.69) and the compactness of the map $G(1,\cdot) : H \to \mathscr{B}(H)$ (see the proof of Theorem 2.2) we have that $\exists \{v_{n_k}(t_{n_k})\}_{k\geq 1} \subset \{v_n(t_n)\}_{n\geq 1}$, $\exists z \in \mathscr{A}$: $v_{n_k}(t_{n_k}) \to z$ in H, $k \to +\infty$. Further, $\forall k \geq 1$ we set $\varphi_k(t) = v_{n_k}(t_{n_k} + t)$, $t \in [0, M]$. Note that $\forall k \geq 1$ $\varphi_k(\cdot) \in \mathscr{D}_{0,M}(v_{n_k}(t_{n_k}))$. Then from Corollary 2.1 there exists a subsequence $\{\varphi_{k_j}\}_{j\geq 1} \subset \{\varphi_k\}_{k\geq 1}$ and an element $\varphi(\cdot) \in \mathscr{D}_{0,M}(z)$:

$$\varphi_{k_j} \to \varphi \text{ in } C([0, M]; H), \quad j \to +\infty. \tag{2.77}$$

At that, taking into account the invariance of $\mathscr{A}$ (see Theorem 2.2), $\forall t \in [0, M]$ $\varphi(t) \in \mathscr{A}$. In consequence of Lemma 2.11 there exist $y(\cdot), v(\cdot) \in \mathscr{K}$: $y(0) = z$, $v(0) = \varphi(M)$. For any $t \in \mathbf{R}$ we set

$$\psi(t) = \begin{cases} y(t), & t \leq 0, \\ \varphi(t), & t \in [0, M], \\ v(t - M), & t \geq M. \end{cases}$$

In view of Lemma 2.7 $\psi(\cdot) \in \mathscr{K}$. Therefore, from (2.76) we have:

$$\forall k \geq 1 \quad \|\Pi_M v_{n_k}(t_{n_k} + \cdot) - \Pi_M \psi(\cdot)\|_{C([0,M];H)} = \|\varphi_k - \varphi\|_{C([0,M];H)} \geq \varepsilon,$$

and we obtain the contradiction with (2.77).

Thus, the set $\mathscr{P}$ constructed in (2.75) is the trajectory attractor in the trajectory space $\mathscr{K}_+$ with respect to the topology of $C^{loc}(\mathbf{R}_+; H)$.

2.4.3 Comments

From results of Sects. 2.4.1 and 2.4.2 it follows that the m-semiflow G, constructed on all weak solutions of (6.5), has the compact invariant global attractor $\mathscr{A}$. For all weak solutions of (6.5), defined on semi-infinite interval $[0, +\infty)$, there exists the trajectory attractor $\mathscr{P}$. At that

$$\mathscr{A} = \mathscr{P}(0) = \{y(0) \mid y \in \mathscr{K}\}, \quad \mathscr{P} = \Pi_+\mathscr{K},$$

where $\mathscr{K}$ is the family of all complete trajectories of differential-operator inclusion (6.5) in $C^{loc}(\mathbf{R}; H) \cap L_{\infty}(\mathbf{R}; H)$. Therefore, the equality of global attractors in the sense of [24, Definition 6, p. 88] as well as [37, Definition 2.2, c. 201] is proved. Questions concerned with connectedness of constructed attractors in the general case are opened. Note that approaches proposed in works [24, 37] are based on properties of solutions for evolutional objects. The approach considered in this work is based on properties of an interaction function A from (6.5) and properties of phase spaces.

2.4.4 Conclusion

We investigated the dynamics as $t \to +\infty$ of all global weak solutions defined on $[0, +\infty)$ for a class of autonomous differential-operator inclusions with pseudomonotone nonlinear dependence between determinative parameters of a problem. We proved the existence of the global compact and compact trajectory attractors, investigated their structure and checked the equality of global attractors in the sense of Definition 6 from [24] as well as in the sense of Definition 2.2 from [37]. Obtained results allows us to study the dynamics of solutions of new classes of evolution equations of nonlinear mathematical models of geophysical and socioeconomical processes and fields with interaction function of pseudomonotone type satisfying the condition of “no more than polynomial growth” and standard sign condition.

2.5 Auxiliary Properties of Solutions for the Second Order Evolution Inclusions and Hemivariational Inequalities for Viscoelastic Processes

Let the next conditions are fulfilled (see Example 1):

(H_1) V, Z, H are Hilbert spaces; $H^* \equiv H$ and we have such chain of dense and compact embeddings:

$$V \subset Z \subset H \equiv H^* \subset Z^* \subset V^*;$$

(H_2) $f_0 \in V^*$;
(A_1) $\exists c > 0 : \forall u \in V, \forall d \in A_0(u)\ \|d\|_{V^*} \leq c(1 + \|u\|_V)$;
(A_2) $\exists \alpha, \beta > 0 : \forall u \in V, \forall d \in A_0(u)\ \langle d, u\rangle_V \geq \alpha\|u\|_V^2 - \beta$;
(A_3) $A_0 = A_1 + A_2$, where $A_1 : V \to V^*$ is linear, selfconjugated, positive operator, $A_2 : V \overset{\to}{\to} V^*$ satisfies such conditions:

(a) There exists such Hilbert space Z, that the embedding $V \subset Z$ is dense and compact one and the embedding $Z \subset H$ is dense and continuous one;

(b) For any $u \in Z$ the set $A_2(u)$ is nonempty, convex and weakly compact one in Z^*;
(c) $A_2 : Z \overset{\to}{\to} Z^*$ is a bounded map, i.e. A_2 converts bounded sets from Z into bounded sets in the space Z^*;
(d) $A_2 : Z \overset{\to}{\to} Z^*$ is a demiclosed map, i.e. if $u_n \to u$ in Z, $d_n \to d$ weakly in Z^*, $n \to +\infty$, and $d_n \in A_2(u_n)$ $\forall n \geq 1$ then $d \in A_2(u)$;

(B_1) $B_0 : V \to V^*$ is a linear selfconjugated operator;

(B_2) $\exists \gamma > 0 : \langle B_0 u, u\rangle_V \geq \gamma \|u\|_V^2$.
Here $\langle\cdot,\cdot\rangle_V : V^* \times V \to \mathbf{R}$ is the duality in $V^* \times V$, coinciding on $H \times V$ with the inner product $(\cdot,\cdot)$ in Hilbert space H.

Note that from (A_1)–(A_3), [25, 42] it follows that the map A_0 satisfies such condition:
$(A_3)'$ $A_0 : V \overset{\to}{\to} V^*$ is (generalized) λ_0-pseudomonotone, i.e.

(a) For any $u \in V$ the set $A_0(u)$ is nonempty, convex and weakly compact one in V^*;
(b) If $u_n \to u$ weakly in V, $n \to +\infty$, $d_n \in A_0(u_n)$ $\forall n \geq 1$ and $\overline{\lim\limits_{n\to\infty}} \langle d_n, u_n - u\rangle_V \leq 0$ then $\forall \omega \in V$ $\exists d(\omega) \in A_0(u)$:

$$\underline{\lim_{n\to+\infty}} \langle d_n, u_n - \omega\rangle_V \geq \langle d(\omega), u - \omega\rangle_V;$$

(c) The map A_0 is upper semicontinuous one that acts from an arbitrary finite-dimensional subspace of V into V^*, endowed with weak topology.

Thus, we investigate the dynamic of all weak solutions of the second order nonlinear autonomous differential-operator inclusion

$$y''(t) + A_0(y'(t)) + B_0(y(t)) \ni f_0, \quad \text{for a.e. } t > 0 \tag{2.78}$$

as $t \to +\infty$, which are defined as $t \geq 0$, where parameters of the problem satisfy conditions (H_1), (H_2), (A_1)–(A_3), (B_1)–(B_2).

As a *weak solution* of the evolution inclusion (2.78) on the interval $[\tau, T]$ we consider such pair of elements $(u(\cdot), u'(\cdot))^T \in L_2(\tau, T; V \times V)$, that for some $d(\cdot) \in L_2(\tau, T; V^*)$

$$d(t) \in A_0(u'(t)) \quad \text{for almost every (a.e.)} \quad t \in (\tau, T),$$

$$-\int_\tau^T (\zeta'(t), u'(t))dt + \int_\tau^T \langle d(t), \zeta(t)\rangle_V dt$$

$$+\int_\tau^T \langle B_0 u(t), \zeta(t)\rangle_V dt = \int_\tau^T \langle f_0, \zeta(t)\rangle_V \quad \forall \zeta \in C_0^\infty([\tau, T]; V), \tag{2.79}$$

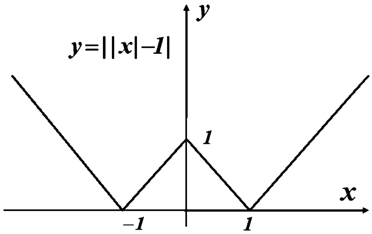

Fig. 2.6 Clarkes subdifferentiable nonconvex functional

where u' is the derivative of the element $u(\cdot)$ in the sense of the space of distributions $\mathscr{D}^*([\tau, T]; V^*)$.

As a *generalized solution* of the problem (3)–(7) we consider the weak solution of the corresponding problem (2.78). This definition is coordinated with Definition 3 from [25].

We have to note that abstract theorems on existence of solutions for such problems as the problem (2.78) and the optimal control problems for weaker conditions for parameters of problems are considered in works [25, 38, 39, 41, 42]. Here we consider Problem 2 from [25], for which we can (as follows from results of the given paper) have not only the abstract result on existence of weak solution but we can investigate the behaviour of all weak solutions as $t \to +\infty$ in the phase space $V \times H$ and study the structure of the global and trajectory attractors. Underline that results concerning multivalued dynamic of displacements and velocities can be applied to hemivariational inequalities with multidimensional superpotential laws (Fig. 2.6).

2.5.1 Preliminary Results

Further, without loss the generality, on the space V we consider the equivalent norm $\|u\|_V = \sqrt{\langle B_0 u, u\rangle_V}$, $u \in V$. The given norm is generated by the inner product $(u, v)_V = \langle B_0 u, v\rangle_V$, $u, v \in V$. For fixed $\tau < T$ let us consider

$$X_{\tau,T} = L_2(\tau, T; V), \quad X^*_{\tau,T} = L_2(\tau, T; V^*), \quad W_{\tau,T} = \{u \in X_{\tau,T} | u' \in X^*_{\tau,T}\},$$

$$A_{\tau,T} : X_{\tau,T} \rightrightarrows X^*_{\tau,T}, \quad \mathscr{A}_{\tau,T}(y) = \{d \in X^*_{\tau,T} | d(t) \in A_0(y(t)) \text{ for a.e. } t \in (\tau, T)\},$$

$$B_{\tau,T} : X_{\tau,T} \to X^*_{\tau,T}, \quad B_{\tau,T}(y)(t) = B_0(y(t)) \text{ for a.e. } t \in (\tau, T),$$

$$f_{\tau,T} \in X^*_{\tau,T}, \quad f_{\tau,T}(t) = f_0 \text{ for a.e. } t \in (\tau, T).$$

Note, that the space $W_{\tau,T}$ is the Hilbert space with the graph norm of the derivative (cf. [41, 42]):

$$\|u\|^2_{W_{\tau,T}} = \|u\|^2_{X_{\tau,T}} + \|u'\|^2_{X^*_{\tau,T}}, \quad u \in W_{\tau,T}. \tag{2.80}$$

From [25, Lemma 7, p. 516], (A_1), (A_2), $(A_3)'$ it follows that $\mathscr{A}_{\tau,T} : X_{\tau,T} \rightrightarrows X^*_{\tau,T}$ satisfies the next conditions:

(N_1) $\exists C_1 > 0$: $\|d\|_{X^*_{\tau,T}} \le C_1(1 + \|y\|_{X_{\tau,T}})$ $\forall y \in X_{\tau,T}$, $\forall d \in \mathscr{A}_{\tau,T}(y)$;
(N_2) $\exists C_2, C_3 > 0$: $\langle d, y\rangle_{X_{\tau,T}} \ge C_2\|y\|^2_{X_{\tau,T}} - C_3$ $\forall y \in X_{\tau,T}$, $\forall d \in \mathscr{A}_{\tau,T}(y)$;
(N_3) $\mathscr{A}_{\tau,T} : X_{\tau,T} \rightrightarrows X^*_{\tau,T}$ is (generalized) w_λ-pseudomonotone on $W_{\tau,T}$, i.e.

(a) For any $y \in X_{\tau,T}$ the set $\mathscr{A}_{\tau,T}(y)$ is a nonempty, convex and weakly compact one in $X^*_{\tau,T}$;
(b) $\mathscr{A}_{\tau,T}$ is the upper semicontinuous map as the map that acts from an arbitrary finite dimensional subspace from $X_{\tau,T}$ into $X^*_{\tau,T}$, endowed by the weak topology;
(c) If $y_n \to y$ weakly in $W_{\tau,T}$, $d_n \in \mathscr{A}_{\tau,T}(y_n)$ $\forall n \ge 1$, $d_n \to d$ weakly in $X^*_{\tau,T}$ and

$$\overline{\lim_{n\to+\infty}} \langle d_n, y_n - y\rangle_{X_{\tau,T}} \le 0$$

then $d \in \mathscr{A}_{\tau,T}(y)$ and $\lim\limits_{n\to+\infty} \langle d_n, y_n\rangle_{X_{\tau,T}} = \langle d, y\rangle_{X_{\tau,T}}$.

Here $\langle\cdot,\cdot\rangle_{X_{\tau,T}} : X^*_{\tau,T} \times X_{\tau,T} \to \mathbf{R}$ is the pairing in $X^*_{\tau,T} \times X_{\tau,T}$ coinciding on $L_2(\tau, T; H) \times X_{\tau,T}$ with the inner product in $L_2(\tau, T; H)$, i.e.

$$\forall u \in L_2(\tau, T; H),\ \forall v \in X_{\tau,T} \quad \langle u, v\rangle_{X_{\tau,T}} = \int_\tau^T (u(t), v(t))dt.$$

Note also (cf. [16, Theorem IV.1.17, P. 177]), that the embedding $W_{\tau,T} \subset C([\tau, T]; H)$ is continuous and dense one, moreover

$$\forall u, v \in W_{\tau,T} \quad (u(T), v(T)) - (u(\tau), v(\tau)) = \int_\tau^T \Big[\langle u'(t), v(t)\rangle_V + \langle v'(t), u(t)\rangle_V\Big]dt. \tag{2.81}$$

From the definition of derivative in the sense of $\mathscr{D}([\tau, T]; V^*)$ and the equality (2.79) it directly follows such statement:

Lemma 2.12. *Each weak solution $(y(\cdot), y'(\cdot))^T$ of the problem (2.78) on the interval $[\tau, T]$ belongs to the space $C([\tau, T]; V) \times W_{\tau,T}$. Moreover*

$$y'' + \mathscr{A}_{\tau,T}(y') + B_{\tau,T}(y) \ni f_{\tau,T}. \tag{2.82}$$

Vice versa, if $y(\cdot) \in C([\tau, T]; V)$, $y'(\cdot) \in W_{\tau,T}$ and $y(\cdot)$ satisfies (2.82), then $(y(\cdot), y'(\cdot))^T$ is a weak solution of (2.78) on $[\tau, T]$.

A weak solution of the problem (2.78) with initial data

$$y(\tau) = a, \quad y'(\tau) = b \tag{2.83}$$

on the interval $[\tau, T]$ exists for any $a \in V$, $b \in H$. It follows from [25, Theorem 11, p. 523]. Thus, the next lemma holds true.

Lemma 2.13. *For any $\tau < T$, $a \in V$, $b \in H$ the Cauchy problem (2.78), (2.83) has a weak solution $(y, y')^T \in X_{\tau,T} \times X_{\tau,T}$. Moreover, each weak solution $(y, y')^T$ of the Cauchy problem (2.78), (2.83) on the interval $[\tau, T]$ belongs to the space $C([\tau, T]; V) \times W_{\tau,T}$ and y satisfies (2.82).*

Remark 2.3. Since $W_{\tau,T} \subset C([\tau, T]; H)$, initial data (2.83) have sense.

Let us consider the next denotations: $E = V \times H$, $\forall \varphi_\tau = (a, b)^T \in E$

$$\mathscr{D}_{\tau,T}(\varphi_\tau) = \left\{ \begin{pmatrix} y(\cdot) \\ y'(\cdot) \end{pmatrix} \middle| \begin{array}{l} (y, y')^T \text{ is a weak solution of (2.78) on } [\tau, T], \\ y(\tau) = a, \ y'(\tau) = b \end{array} \right\}.$$

From Lemma 2.13 it follows that $\mathscr{D}_{\tau,T}(\varphi_\tau) \subset C([\tau, T]; V) \times W_{\tau,T} \subset C([\tau, T]; E)$.

Let us complete the given subsection by checking that translation and concatenation of weak solutions is a weak solution too.

Lemma 2.14. *If $\tau < T$, $\varphi_\tau \in E$, $\varphi(\cdot) \in \mathscr{D}_{\tau,T}(\varphi_\tau)$, then $\psi(\cdot) = \varphi(\cdot + s) \in \mathscr{D}_{\tau-s,T-s}(\varphi_\tau)$ $\forall s$. If $\tau < t < T$, $\varphi_\tau \in E$, $\varphi(\cdot) \in \mathscr{D}_{\tau,t}(\varphi_\tau)$ and $\psi(\cdot) \in \mathscr{D}_{t,T}(\varphi_\tau)$, then*

$$\theta(s) = \begin{cases} \varphi(s), & s \in [\tau, t], \\ \psi(s), & s \in [t, T] \end{cases}$$

belongs to $\mathscr{D}_{\tau,T}(\varphi_\tau)$.

Proof. The first part of the statement of this lemma follows from the autonomy of the inclusion (2.78). The proof of the second part follows from the definition of the solution of (2.79), Lemma 2.12 and from that fact that $z \in W_{\tau,T}$ as soon as $v \in W_{\tau,t}$, $u \in W_{t,T}$ and $v(t) = u(t)$, where

$$z(s) = \begin{cases} v(s), & s \in [\tau, t], \\ u(s), & s \in [t, T] \end{cases}$$

For the proof of the last we can use the definition of the derivative in the sense $\mathscr{D}([\tau, T]; V^*)$ and the formula (2.81).

2.5.2 *Auxiliary Properties of the Resolving Operator*

As a rule the proof of the existence of the compact global and trajectory attractors for evolution inclusions and, in particular, inclusions like (2.78) is based on properties of family of weak solutions of the problem (2.78) connected to absorbtion of the generated m-semiflow, closedness of its graph and its asymptotic compactness (cf. [21, 24, 36, 37] and references therein). The next lemma on a priory estimates and Theorem 2.4 on dependence of solutions on initial data play the key part when investigating the dynamic of solutions of the problem (2.78) as $t \to +\infty$.

Lemma 2.15. *There exists constants $c_1, c_2, c_3, c_4 > 0$ such that for any finite interval $[\tau, T]$ and for each weak solution $(y, y')^T$ of the problem (2.78) on $[\tau, T]$ the next estimates holds true: $\forall t \geq s$, $t, s \in [\tau, T]$*

$$\begin{aligned} &\|y'(t)\|_H^2 + \|y(t)\|_V^2 + \alpha \int_s^t \|y'(\xi)\|_V^2 d\xi \\ &\quad \leq \|y'(s)\|_H^2 + \|y(s)\|_V^2 + c_4(t-s)(\|f\|_{V^*}^2 + 1), \end{aligned} \tag{2.84}$$

$$\|y'(t)\|_H^2 + \|y(t)\|_V^2 \leq c_1(\|y'(s)\|_H^2 + \|y(s)\|_V^2)e^{-c_2(t-s)} + c_3(1 + \|f\|_{V^*}^2). \tag{2.85}$$

Proof. The inequality (2.84) obviously follows from Lemma 2.12 and Condition (A_2).

Let us prove now (2.85). We fix an arbitrary finite interval $[\tau, T]$ and an arbitrary weak solution $(y, y')^T$ of the problem (2.78) on $[\tau, T]$. Note that $y \in C([\tau, T]; V)$, $y' \in W_{\tau,T}$. For any $t \in [\tau, T]$ let us set

$$Y(t) = \frac{1}{2}\|y'(t)\|_H^2 + \frac{1}{2}\|y(t)\|_V^2 + \varepsilon(y'(t), y(t)),$$

where $\varepsilon = \frac{2\lambda_1\alpha}{5+2\lambda_1 c^2} > 0$, $\lambda_1 > 0$ such that the next inequality takes place:

$$\lambda_1\|u\|_H^2 \leq \|u\|_V^2 \quad \forall u \in V. \tag{2.86}$$

Firstly we check the next inequality

$$\frac{dY(t)}{dt} \leq -\alpha_1 Y(t) + \alpha_2 \quad \text{for a.e. } t \in (\tau, T), \tag{2.87}$$

where $\alpha_1 = \frac{\varepsilon\sqrt{\lambda_1}}{2(\varepsilon+2\sqrt{\lambda_1})} > 0$, $\alpha_2 = \beta + 2\varepsilon c^2 + \|f\|_{V^*}^2(\frac{1}{2\alpha} + 2\varepsilon) > 0$.

From Conditions (A_1), (A_2) and the definition of a weak solution of the problem (2.78) on $[\tau, T]$ we have:

$$\begin{aligned} \frac{dY(t)}{dt} &= (y''(t), y'(t)) + \langle B_0 y(t), y'(t)\rangle_V + \varepsilon(y''(t), y(t)) + \varepsilon\|y'(t)\|_H^2 \\ &\leq -\alpha\|y'(t)\|_V^2 - \varepsilon\|y(t)\|_V^2 + \varepsilon\|y'(t)\|_H^2 + \|f\|_{V^*}\|y'(t)\|_V \\ &\quad + \varepsilon\|y(t)\|_V(c + \|f\|_{V^*}) + \varepsilon c\|y'(t)\|_V\|y(t)\|_V + \beta. \end{aligned} \tag{2.88}$$

Note that

$$c\|y'(t)\|_V\|y(t)\|_V \le \frac{c^2}{2}\|y'(t)\|_V^2 + \frac{1}{2}\|y(t)\|_V^2,$$

$$\|y(t)\|_V(c+\|f\|_{V^*}) \le \frac{\|y(t)\|_V^2}{4} + (c+\|f\|_{V^*})^2 \le \frac{\|y(t)\|_V^2}{4} + 2c^2 + 2\|f\|_{V^*}^2,$$

$$\|f\|_{V^*}\|y'(t)\|_V \le \frac{\alpha\|y'(t)\|_V^2}{2} + \frac{\|f\|_{V^*}^2}{2\alpha}.$$

Applying considered inequalities to the right part of (2.88), by the help of (2.86) we obtain:

$$\frac{dY(t)}{dt} \le -\frac{\varepsilon}{4}(\|y'(t)\|_H^2 + \|y(t)\|_V^2) + \beta + 2\varepsilon c^2 + \|f\|_{V^*}^2\left(\frac{1}{2\alpha} + 2\varepsilon\right). \quad (2.89)$$

Note that

$$|(y'(t), y(t))| \le \frac{1}{2\sqrt{\lambda_1}}(\|y'(t)\|_H^2 + \|y(t)\|_V^2). \quad (2.90)$$

Therefore, from inequalities (2.89), (2.90) we have (2.87).

From (2.87) and Gronwall-Bellman lemma we obtain

$$\forall \tau \le s \le t \le T \quad Y(t) \le Y(s)e^{-\alpha_1(t-s)} + \frac{\alpha_2}{\alpha_1}(1+\|f\|_{V^*}^2).$$

Thus, in view of (2.90), the next inequality takes place:

$$\forall t \in [\tau, T] \quad \|y'(t)\|_H^2 + \|y(t)\|_V^2$$

$$\le \frac{\sqrt{\lambda_1}+\varepsilon}{\sqrt{\lambda_1}-\varepsilon}\left((\|y'(s)\|_H^2 + \|y(s)\|_V^2)e^{-\alpha_1(t-s)} + \frac{\alpha_2}{\alpha_1}(1+\|f\|_{V^*}^2)\right).$$

Note that $\sqrt{\lambda_1} > \varepsilon$, in view of $\alpha \le c$ and $2\lambda^2 - 2\lambda + 5 \ge 0$ $\forall \lambda \in \mathbf{R}$, in particular for $\lambda = \sqrt{\lambda_1}c$.

If we set $c_1 = \frac{\sqrt{\lambda_1}+\varepsilon}{\sqrt{\lambda_1}-\varepsilon} > 0$, $c_2 = \alpha_1$, $c_3 = \frac{\alpha_2}{\alpha_1}\cdot c_1 > 0$, we obtain the necessary inequality.

Theorem 2.4. *Let* $\tau < T$, $\{(u_n, u_n')^T\}_{n\ge1}$ *be an arbitrary sequence of weak solutions of (2.78) on* $[\tau, T]$ *such that* $u_n(\tau) \to u_\tau$ *weakly in* V, $u_n'(\tau) \to u_\tau'$ *weakly in* H. *Then there exist* $\{(u_{n_k}, u_{n_k}')^T\}_{k\ge1} \subset \{(u_n, u_n')^T\}_{n\ge1}$ *and* $(u, u')^T \in \mathscr{D}_{\tau,T}((u_\tau, u_\tau')^T)$ *such that*

$$\forall \varepsilon \in (0, T-\tau) \ \max_{t\in[\tau+\varepsilon,T]}\|u_{n_k}'(t) - u'(t)\|_H \to 0, \ k \to +\infty, \quad (2.91)$$

$$u_{n_k}(t) \to u(t) \textit{ weakly in } V, \textit{ uniformly on } [\tau, T], \ k \to +\infty. \quad (2.92)$$

If supplementary $(u_n(\tau), u_n'(\tau))^T \to (u_\tau, u_\tau')^T$ *in* E, $n \to +\infty$, *then* $(u_{n_k}(\cdot), u_{n_k}'(\cdot))^T \to (u(\cdot), u'(\cdot))^T$ *in* $C([\tau, T]; E)$, $k \to +\infty$.

Proof. Under conditions of the theorem in view of Lemma 2.12, for any $n \geq 1$ $\varphi_n = (u_n, u_n')^T \in C([\tau, T]; E)$. Moreover, from Lemma 2.13, 2.15 we obtain that

$$\begin{aligned}
&\forall n \geq 1\ \exists d_n \in \mathscr{A}_{\tau,T}(u_n') : \\
&u_n''(t) + d_n(t) + B_0 u_n(t) = f \text{ for a.e. } t \in (\tau, T);
\end{aligned} \tag{2.93}$$

$$\begin{aligned}
&\exists C > 0 :\ \forall n \geq 1\ \|u_n'\|_{X_{\tau,T}} + \|u_n''\|_{X_{\tau,T}^*} \\
&+ \|u_n'\|_{C([\tau,T];H)} + \|d_n\|_{X_{\tau,T}^*} + \|u_n\|_{C([\tau,T];V)} \leq C.
\end{aligned} \tag{2.94}$$

Note that $\forall n \geq 1$ $\forall t \in [\tau, T]$ $u_n(t) = v_n(t) + u_{\tau,n}$, where $v_n(t) = \int\limits_\tau^t u_n'(s)ds$, $(u_{\tau,n}, u_{\tau,n}')^T = \varphi_{\tau,n}$. At that

$$\forall n \geq 1,\ \forall t, s \in [\tau, T]\ \|v_n(t) - v_n(s)\|_V \leq C|t - s|^{\frac{1}{2}},\ v_n(0) = \bar{0}. \tag{2.95}$$

Therefore, from (2.93)–(2.95), continuity of the embedding $W_{\tau,T} \subset C([\tau, T]; H)$, compactness of the embedding $W_{\tau,T} \subset L_2(\tau, T; H)$, reflexivity of spaces $W_{\tau,T}$, $X_{\tau,T}$, $X_{\tau,T}^*$ we have that up to a subsequence $\{u_{n_k}, d_{n_k}\}_{k\geq 1} \subset \{u_n, d_n\}_{n\geq 1}$ for some $u \in C([\tau, T]; V)$, $u' \in W_{\tau,T}$, $d \in X_{\tau,T}^*$ the next convergences take place:

$$\begin{aligned}
&v_{n_k} \to v \text{ in } C([\tau, T]; V),\ \ u_{n_k}(t) \to u(t) \text{ weakly in } V\ \forall t \in [\tau, T], \\
&u_{n_k}' \to u' \text{ weakly in } X_{\tau,T},\ u_{n_k}'' \to u'' \text{ weakly in } X_{\tau,T}^*, \\
&d_{n_k} \to d \text{ weakly in } X_{\tau,T}^*,\ u_{n_k}' \to u' \text{ weakly in } C([\tau, T]; H), \\
&u_{n_k}' \to u' \text{ in } L_2(\tau, T; H),\ u_{n_k}'(t) \to u'(t) \text{ in } H \text{ for a.e. } t \in (\tau, T),\ k \to +\infty,
\end{aligned} \tag{2.96}$$

where $v(\cdot) = u(\cdot) - u_\tau$. Let us complete the proof of the theorem in several steps.

Step 1. Show that

$$u'' = f_{\tau,T} - d - B_{\tau,T}(u). \tag{2.97}$$

Indeed, $\forall k \geq 1$, $\forall \zeta \in C_0^\infty([\tau, T]; V)$

$$\begin{aligned}
&-\langle \zeta', u_{n_k}' \rangle_{X_{\tau,T}} + \langle d_{n_k}, \zeta \rangle_{X_{\tau,T}} + \langle B_{\tau,T}(v_{n_k}), \zeta \rangle_{X_{\tau,T}} \\
&+ \int\limits_\tau^T \langle B_0 u_{n_k,\tau}, \zeta(t) \rangle_V dt = \langle f_{\tau,T}, \zeta \rangle_{X_{\tau,T}}.
\end{aligned} \tag{2.98}$$

Further, let us pass in (2.98) to the limit as $k \to +\infty$. We obtain:

$$\forall \zeta \in C_0^\infty([\tau,T];V) \quad -\langle \zeta', u'\rangle_{X_{\tau,T}} + \langle d,\zeta\rangle_{X_{\tau,T}} + \langle B_{\tau,T}(v),\zeta\rangle_{X_{\tau,T}} + \int_\tau^T \langle B_0 u_\tau, \zeta(t)\rangle_V dt = \langle f_{\tau,T},\zeta\rangle_{X_{\tau,T}}.$$

Thus, using properties of Bochner's integral, $\forall \varphi \in C_0^\infty([\tau,T])\ \forall h \in V$

$$\begin{aligned}
-\left(\int_\tau^T u'(s)\varphi'(s)ds, h\right)_H &= -\int_\tau^T (h, u'(s))_H \varphi'(s)ds \\
&= \int_\tau^T \langle f - d(s) - B_0 v(s) - B_0 u_\tau, h\rangle_H \varphi(s)ds \\
&= \left\langle \int_\tau^T [f_{\tau,T}(s) - d(s) - B_{\tau,T}(u)(s)]\varphi(s)ds, h\right\rangle_V .
\end{aligned}$$

Finally, the relation (2.97) follows from the definition of derivative of an element u' in the sense $\mathscr{D}^*([\tau,T];V^*)$.

Step 2. From (2.96) it follows that $\exists \{\varepsilon_j\}_{j\ge 1} \subset (\tau,T)$:

$$\varepsilon_j \searrow 0+,\ j \to +\infty,\quad \forall j \ge 1\ u'_{n_k}(\tau+\varepsilon_j) \to u'(\tau+\varepsilon_j) \text{ in } H,\ k \to +\infty. \tag{2.99}$$

Let us fix an arbitrary $\varepsilon \in \{\varepsilon_j\}_{j\ge 1}$ and show that

$$d(t) \in A_0(u'(t)) \text{ for a.e. } t \in (\tau+\varepsilon, T), \tag{2.100}$$

using pseudomonotony of $\mathscr{A}_{\tau+\varepsilon,T}$ on $W_{\tau+\varepsilon,T}$.

Let us consider restrictions $u_{n_k}(\cdot)$, $d_{n_k}(\cdot)$, $u(\cdot)$, $d(\cdot)$ on the interval $[\tau+\varepsilon,T]$. For the simplicity of suggestions we denote these restrictions by the same symbols: $u_{n_k}(\cdot)$, $d_{n_k}(\cdot)$, $u(\cdot)$, $d(\cdot)$ correspondingly. From (2.96) it follows that

$$u'_{n_k} \to u' \text{ weakly in } W_{\tau+\varepsilon,T},\ d_{n_k} \to d \text{ weakly in } X^*_{\tau+\varepsilon,T},\ k \to +\infty. \tag{2.101}$$

Show that

$$\overline{\lim_{k\to+\infty}} \langle d_{n_k}, u'_{n_k} - u'\rangle_{X_{\tau+\varepsilon,T}} \le 0 \tag{2.102}$$

Indeed,

$$
\begin{aligned}
\forall k \geq 1 \quad & \int_{\tau+\varepsilon}^{T} \langle d_{n_k}(s), u'_{n_k}(s) - u'(s)\rangle_V ds \\
&= \int_{\tau+\varepsilon}^{T} \langle f, u'_{n_k}(s) - u'(s)\rangle_V ds + \int_{\tau+\varepsilon}^{T} \langle u''_{n_k}(s), u'(s) - u'_{n_k}(s)\rangle_V ds \\
&\quad + \int_{\tau+\varepsilon}^{T} \langle B_0 u_{\tau,n_k}, u'(s) - u'_{n_k}(s)\rangle_V ds \\
&\quad + \int_{\tau+\varepsilon}^{T} \langle B_0 v_{n_k}, u'(s) - u'_{n_k}(s)\rangle_V ds \\
&:= I_{1,k} + I_{2,k} + I_{3,k} + I_{4,k}.
\end{aligned} \tag{2.103}
$$

From (2.101) it follows that

$$
I_{1,k} \to 0, \quad k \to +\infty. \tag{2.104}
$$

In consequence of (2.81), (2.96) and (2.99) we obtain that

$$
\begin{aligned}
\forall k \geq 1 \ I_{2,k} &= \int_{\tau+\varepsilon}^{T} \langle u''_{n_k}(s), u'(s)\rangle_V ds - \frac{1}{2}(\|u'_{n_k}(T)\|_H^2 - \|u'_{n_k}(\tau+\varepsilon)\|_H^2), \\
\overline{\lim_{k\to+\infty}} I_{2,k} &\leq \int_{\tau+\varepsilon}^{T} \langle u''(s), u'(s)\rangle_V ds - \frac{1}{2}(\|u'(T)\|_H^2 - \|u'(\tau+\varepsilon)\|_H^2) = 0.
\end{aligned} \tag{2.105}
$$

In view of (2.95), (2.96) and properties of Bochner's integral we have that $\forall k \geq 1$

$$
I_{3,k} = \langle B_0 u_{\tau,n_k}, v(T) - v(\tau+\varepsilon) - v_{n_k}(T) + v_{n_k}(\tau+\varepsilon)\rangle_V \to 0, \ k \to +\infty. \tag{2.106}
$$

$$
\begin{aligned}
|I_{4,k}| &\leq \left| \int_{\tau+\varepsilon}^{T} \langle B_0 v(s), u'(s) - u'_{n_k}(s)\rangle_V ds \right| \\
&+ \|B_0\|_{\mathscr{L}(V;V^*)} \|v_{n_k} - v\|_{C([\tau,T];V)} \cdot 2C \cdot (T - \tau - \varepsilon)^{\frac{1}{2}} \to 0, \ k \to +\infty.
\end{aligned} \tag{2.107}
$$

Thus, if we pass in (2.103) to the upper limit as $k \to +\infty$, in view of (2.104)–(2.107), we obtain (2.102).

Further, due to (2.93), (2.101), (2.102) and pseudomonotony of $\mathscr{A}_{\tau+\varepsilon,T}$ on $W_{\tau+\varepsilon,T}$ we obtain (2.100).

Step 3. From (2.99), an arbitrariness of $\varepsilon \in \{\varepsilon_j\}_{j\geq 1}$, the relation (2.100) and definition of $\mathscr{A}_{\tau,T}(u')$ we obtain that $(u, u')^T \in \mathscr{D}_{\tau,T}((u_\tau, u'_\tau)^T)$.

Step 4. From (2.96) it directly follows (2.92).

Step 5. Let us check (2.91) using the method by contradiction. We suggest that $\exists\varepsilon > 0, \exists L > 0, \exists\{u_{k_j}\}_{j\geq 1} \subset \{u_{n_k}\}_{k\geq 1}$:

$$\forall j \geq 1 \ \max_{[\tau,T]} \|u'_{k_j}(t) - u'(t)\|_H = \|u'_{k_j}(t_j) - u'(t_j)\|_H \geq L.$$

Without loss of the generality we can guess that $t_j \to t_0 \in [\tau, T]$, $j \to +\infty$. Therefore, in view of continuity of $u' : [\tau, T] \to H$,

$$\varliminf_{j\to+\infty} \|u'_{k_j}(t_j) - u'(t_0)\|_H \geq L. \tag{2.108}$$

On the other hand we show that

$$u'_{k_j}(t_j) \to u'(t_0) \text{ in } H, \ j \to +\infty. \tag{2.109}$$

Step 5.1. Firstly we prove that

$$u'_{k_j}(t_j) \to u'(t_0) \text{ weakly in } H, \ j \to +\infty. \tag{2.110}$$

For a fixed $h \in V$ from (2.96) it follows that the sequence of real functions $(u'_{n_k}(\cdot), h) : [\tau, T] \to \mathbf{R}$ is uniformly bounded and equipotentionally continuous one. Taking into account (2.96) and density of the embedding $V \subset H$ we obtain that $u'_{n_k}(t) \to u'(t)$ weakly in H uniformly on $[\tau, T]$, $k \to +\infty$, whence it follows (2.110).

Step 5.2. Let us prove that

$$\varlimsup_{j\to+\infty} \|u'_{k_j}(t_j)\|_H \leq \|u'(t_0)\|_H. \tag{2.111}$$

Note that in view of (2.93) and Condition (A_2) we obtain that

$$\begin{aligned}\forall j \geq 1 \text{ for a.e. } t \in (\tau, T) \ &\frac{d}{dt}(\|u'_{k_j}(t)\|^2_H + 2\langle B_0 u_{\tau,k_j}, v_{k_j}(t)\rangle_V \\ &+\|v_{k_j}(t)\|^2_V) \\ = \frac{d}{dt}(\|u'_{k_j}(t)\|^2_H + \|u_{k_j}(t)\|^2_V) &\leq \beta + \frac{\|f\|^2_{V^*}}{4\alpha} =: \bar{\beta}.\end{aligned}$$

Similarly,

$$\text{for a.e. } t \in (\tau, T) \ \frac{d}{dt}(\|u'(t)\|_H^2 + 2\langle B_0 u_\tau, v(t)\rangle_V + \|v(t)\|_V^2) \le \bar{\beta}.$$

Thus, real functions $\{J_j : [\tau, T] \to \mathbf{R} \mid j \ge 0\}$,

$$J_j(t) = \|u'_{k_j}(t)\|_H^2 + \|v_{k_j}(t)\|_V^2 + 2\langle B_0 u_{\tau,k_j}, v_{k_j}(t)\rangle_V - \bar{\beta}t, \tag{2.112}$$

$$J_0(t) = \|u'(t)\|_H^2 + \|v(t)\|_V^2 + 2\langle B_0 u_\tau, v(t)\rangle_V - \bar{\beta}t, \ t \in [\tau, T], \tag{2.113}$$

are steadily nonincreasing, continuous and in view of (2.96),

$$\text{for a.e. } t \in (\tau, T) \ J_j(t) \to J_0(t), \ j \to +\infty. \tag{2.114}$$

Let us fix an arbitrary $\varepsilon_1 > 0$. From (2.114) and continuity of J_0 it follows that

$$\exists \bar{t} \in (\tau, t_0): \ J_j(\bar{t}) \to J_0(\bar{t}), \ j \to +\infty \text{ and } |J_0(\bar{t}) - J_0(t_0)| < \varepsilon_1.$$

Then for rather big $j \ge 1$ $J_j(t_j) - J_0(t_0) \le J_j(\bar{t}) - J_0(\bar{t}) + |J_0(\bar{t}) - J_0(t_0)| < |J_j(\bar{t}) - J_0(\bar{t})| + \varepsilon_1$. From the arbitrariness of $\varepsilon_1 > 0$ we have $\overline{\lim\limits_{j\to+\infty}} J_j(t_j) \le J_0(t_0)$. Hence, taking into account (2.96), we obtain (2.111).

Step 5.3. Equation (2.109) directly follows from (2.110), (2.111) and [16].

Step 5.4. For the completeness of the proof of (2.91) note that (2.109) contradicts with (2.108). Therefore, the validity of (2.91) is checked.

Step 6. Supplementary we suggest that

$$(u_n(\tau), u'_n(\tau))^T \to (u_\tau, u'_\tau)^T \text{ in } E, \ k \to +\infty. \tag{2.115}$$

Step 6.1. From (2.115) and (2.96) it directly follows that

$$u_{n_k} \to u \text{ in } C([\tau, T], V), \ k \to +\infty. \tag{2.116}$$

Step 6.2. For the completeness of this proof it remains to check that

$$u'_{n_k} \to u' \text{ in } C([\tau, T]; H). \tag{2.117}$$

Let us check (2.117) using the method by contradiction. We suggest that $\exists L_1 > 0$, $\exists \{u_{k_j}\}_{j\ge1} \subset \{u_{n_k}\}_{k\ge1}$

$$\forall j \ge 1 \ \|u'_{k_j} - u'\|_{C([\tau,T];H)} = \|u'_{k_j}(t_j) - u'(t_j)\|_H \ge L_1. \tag{2.118}$$

Repeating upper considered suggestions from Step 5 of the proof, taking into account (2.91), without loss of the generality, we can guess that

$$t_j \to \tau,\ u'_{k_j}(t_j) \to u'(\tau) \text{ weakly in } H,\ j \to +\infty;$$
$$\underline{\lim}_{j\to+\infty} \|u'_{k_j}(t_j) - u'(\tau)\|_H \geq L_1 \tag{2.119}$$

Let us consider a sequence of steadily non-decreasing continuous functions $\{J_j\}_{j\geq 0}$, defined in (2.112), (2.113). Since $\forall j \geq 1$ $J_j(t_j) - J_0(\tau) \leq J_j(\tau) - J_0(\tau)$, then in view of (2.96) we obtain that $\overline{\lim}_{j\to+\infty} J_j(t_j) \leq J_0(\tau)$ and, therefore, $\overline{\lim}_{j\to+\infty} \|u'_{k_j}(t_j)\|_H \leq \|u'(\tau)\|_H$. The last inequality together with (2.119) contradicts with (2.118). The theorem is proved.

2.6 Auxiliary Properties of Solutions for the Second Order Evolution Inclusions and Hemivariational Inequalities for Piezoelectric Fields

Now we consider a mathematical model which describes the contact between a piezoelectric body and a foundation (see Example 2). For evolution triple $(V; H; V^*)$, linear operators $R : H \to H$, $G : V \to V^*$ and locally Lipschitz functional $J : H \to \mathbf{R}$ we consider a problem of investigation of dynamics for all weak solutions defined for $t \geq 0$ of non-linear second order autonomous differential-operator inclusion:

$$u''(t) + Ru'(t) + Gu(t) + \partial J(u(t)) \ni \bar{0} \quad \text{a.e.} \quad t > 0. \tag{2.120}$$

We need the following hypotheses:.

$\underline{H(R)}$ $R : H \to H$ is a linear symmetric such that $\exists \gamma > 0 : (Rv, v)_H = \gamma \|v\|_H^2$ $\forall v \in H$;

$\underline{H(G)}$ $G : V \to V^*$ is linear, symmetric and $\exists c_G > 0 : \langle Gv, v\rangle_V \geq c_G \|v\|_V^2$ $\forall v \in V$;

$\underline{H(J)}$ $J : H \to \mathbf{R}$ is a function such that

(i) $J(\cdot)$ is locally Lipschitz and regular [12], i.e.

- For any $x, v \in H$, the usual one-sided directional derivative $J'(x; v) = \lim_{t\searrow 0} \frac{J(x+tv)-J(x)}{t}$ exists,
- For all $x, v \in H$, $J'(x; v) = J^\circ(x; v)$, where $J^\circ(x; v) = \overline{\lim}_{y\to x, t\searrow 0} \frac{J(y+tv)-J(y)}{t}$;

(ii) $\exists c_1 > 0$: $\|\partial J(v)\|_+ \leq c_1(1 + \|v\|_H)$ $\forall v \in H$;

(iii) $\exists c_2 > 0$:
$$[\partial J(v), v]_- \geq -\lambda \|v\|_H^2 - c_2 \quad \forall v \in H,$$

where $\partial J(v) = \{p \in H \mid (p, w)_H \le J^\circ(v; w)\ \forall w \in H\}$ denotes the Clarke subdifferential of $J(\cdot)$ at a point $v \in H$ (see [12] for details), $\lambda \in (0, \lambda_1)$, $\lambda_1 > 0$: $c_G\|v\|_V^2 \ge \lambda_1\|v\|_H^2\ \forall v \in V$;

($\underline{H_0}$) V is a Hilbert space.

The phase space for Problem (2.120) we define Hilbert space $E = V \times H$. Let $-\infty < \tau < T < +\infty$.

Definition 2.1. The function $(u(\cdot), u'(\cdot))^T \in L_\infty(\tau, T; E)$ is called a *weak solution* for (2.120) on (τ, T), if there exists $d \in L_2(\tau, T; H)$, $d(t) \in \partial J(u(t))$ for a.e. $t \in (\tau, T)$, such that $\forall \psi \in V$, $\forall \eta \in C_0^\infty(\tau, T)$

$$-\int_\tau^T (u'(t), \psi)_H \eta'(t)dt + \int_\tau^T \left[(u'(t), \psi)_H + (u(t), \psi)_H + (d(t), \psi)_H\right]\eta(t)dt = 0,$$

We consider a class of functions $W_\tau^T = C([\tau, T]; E)$. Further $\gamma, c_1, c_2, \lambda, \lambda_1$ we recall parameters of Problem (2.120). The main purpose of this work is to investigate the long-time behavior (as $t \to +\infty$) of all weak solutions for the problem (2.120).

To simplify our conclusions from Conditions $H(G)$, $H(R)$ we suppose that

$$(u, v)_V = \langle Gu, v\rangle_V,\ \|v\|_V^2 = \langle Gu, v\rangle_V,\ c_G = 1,\ \gamma(u, v)_H = (Ru, v)_H,\ \gamma\|v\|_H^2 \\ = (Rv, v)_H\ \forall u, v \in V. \tag{2.121}$$

Lebourgues mean value theorem [12, Chap. 2] provides the existence of constants $c_3, c_4 > 0$ and $\mu \in (0, \lambda_1)$:

$$|J(u)| \le c_3(1 + \|u\|_H^2),\quad J(u) \ge -\frac{\mu}{2}\|u\|_H^2 - c_4 \quad \forall u \in H. \tag{2.122}$$

Lemma 2.16. *Let $J : H \to \mathbf{R}$ be a locally Lipschitz and regular functional, $y \in C^1([\tau, T]; H)$. Then for a.e. $t \in (\tau, T)$ $\exists \frac{d}{dt}(J \circ y)(t) = (p, y'(t))\ \forall p \in \partial J(y(t))$. Moreover, $\frac{d}{dt}(J \circ y)(\cdot) \in L_1(\tau, T)$.*

Proof. Since $y \in C^1([\tau, T]; H)$ then y is strictly differentiable at the point t_0 for any $t_0 \in (\tau, T)$. Hence, taking into account the regularity of J and [12, Theorem 2.3.10], we obtain that the functional $J \circ y$ is regular one at $t_0 \in (\tau, T)$ and $\partial(J \circ y)(t_0) = \{(p, y'(t_0)) \mid p \in \partial J(y(t_0))\}$. On the other hand, since $y \in C([\tau, T]; H)$, $J : H \to \mathbf{R}$ is locally Lipschitz then $J \circ y : [\tau, T] \to \mathbf{R}$ is globally Lipschitz and therefore it is absolutely continuous. Hence for a.e. $t \in (\tau, T)$ $\exists \frac{d(J\circ y)(t)}{dt}$, $\frac{d(J\circ y)(\cdot)}{dt} \in L_1(\tau, T)$ and $\int_s^t \frac{d}{d\xi}(J \circ y)(\xi)d\xi = (J \circ y)(t) - (J \circ y)(s)\ \forall \tau \le s < t \le T$. At that taking into account the regularity of $J \circ y$, note that $(J \circ y)^\circ(t_0, v) = (J \circ y)'(t_0, v) = \frac{d(J\circ y)(t_0)}{dt} \cdot v$ for a.e. $t_0 \in (\tau, T)$, $\forall v \in \mathbf{R}$. This implies that for a.e. $t_0 \in (\tau, T)$ $\partial(J \circ y)(t_0) = \left\{\frac{d(J\circ y)(t_0)}{dt}\right\}$.

A weak solution of the problem (2.120) with initial data

$$u(\tau) = a, \quad u'(\tau) = b \tag{2.123}$$

on the interval $[\tau, T]$ exists for any $a \in V, b \in H$. It follows from [23, Theorem 1.4]. Thus, the next lemma holds true (see [33, Lemma 4.1, p. 78] and [33, Lemma 3.1, p. 71]).

Lemma 2.17. *For any $\tau < T, a \in V, b \in H$ the Cauchy problem (2.120), (2.123) has a weak solution $(y, y')^T \in L_\infty(\tau, T; E)$. Moreover, each weak solution $(y, y')^T$ of the Cauchy problem (2.120), (2.123) on the interval $[\tau, T]$ belongs to the space $C([\tau, T]; E)$ and $y'' \in L_2(\tau, T; V^*)$.*

Let us consider the next denotations: $\forall \varphi_\tau = (a, b)^T \in E$ we consider $\mathscr{D}_{\tau,T}(\varphi_\tau) = \{\, (u(\cdot), u'(\cdot))^T \mid (u, u')^T$ is a weak solution of (2.120) on $[\tau, T]$, $u(\tau) = a, u'(\tau) = b \,\}$. From Lemma 2.17 it follows that $\mathscr{D}_{\tau,T}(\varphi_\tau) \subset C([\tau, T]; E) = W_\tau^T$. Let us complete the given subsection by checking that translation and concatenation of weak solutions is a weak solution too.

Lemma 2.18. *If $\tau < T$, $\varphi_\tau \in E$, $\varphi(\cdot) \in \mathscr{D}_{\tau,T}(\varphi_\tau)$, then $\psi(\cdot) = \varphi(\cdot + s) \in \mathscr{D}_{\tau-s,T-s}(\varphi_\tau)$ $\forall s$. If $\tau < t < T$, $\varphi_\tau \in E$, $\varphi(\cdot) \in \mathscr{D}_{\tau,t}(\varphi_\tau)$ and $\psi(\cdot) \in \mathscr{D}_{t,T}(\varphi_\tau)$, then $\theta(s) = \begin{cases} \varphi(s), & s \in [\tau, t], \\ \psi(s), & s \in [t, T] \end{cases}$ belongs to $\mathscr{D}_{\tau,T}(\varphi_\tau)$.*

Proof. The proof is trivial.

Let $\varphi = (a, b)^T \in E$ and

$$\mathscr{V}(\varphi) = \frac{1}{2}\|\varphi\|_E^2 + J(a). \tag{2.124}$$

Lemma 2.19. *Let $\tau < T$, $\varphi_\tau \in E$, $\varphi(\cdot) = (y(\cdot), y'(\cdot))^T \in \mathscr{D}_{\tau,T}(\varphi_\tau)$. Then $\mathscr{V} \circ \varphi : [\tau, T] \to \mathbf{R}$ is absolutely continuous and for a.e. $t \in (\tau, T)$ $\frac{d}{dt}\mathscr{V}(\varphi(t)) = -\gamma\|y'(t)\|_H^2$.*

Proof. Let $-\infty < \tau < T < +\infty$, $\varphi(\cdot) = (y(\cdot), y'(\cdot))^T \in W_\tau^T$ be an arbitrary weak solution of (2.120) on (τ, T). As $\partial J(y(\cdot)) \subset L_2(\tau, T; H)$ then from [33, Lemma 4.1, p. 78] and [33, Lemma 3.1, p. 71] we get that the function $t \to \|y'(t)\|_H^2 + \|y(t)\|_V^2$ is absolutely continuous and for a.e. $t \in (\tau, T)$

$$\frac{1}{2}\frac{d}{dt}\left[\|y'(t)\|_H^2 + \|y(t)\|_V^2\right] = (y''(t) + Gy(t), y'(t))_H = -\gamma\|y'(t)\|_H^2 \\ -(d(t), y'(t))_H, \tag{2.125}$$

where $d(t) \in \partial J(y(t))$ for a.e. $t \in (\tau, T)$ and $d(\cdot) \in L_2(\tau, T; H)$. As $y(\cdot) \in C^1([\tau, T]; H)$ and $J : H \to \mathbf{R}$ is regular and locally Lipschitz, due to Lemma 2.16 we obtain that for a.e. $t \in (\tau, T)$ $\exists \frac{d}{dt}(J \circ y)(t)$. Moreover, $\frac{d}{dt}(J \circ y)(\cdot) \in L_1(\tau, T)$

and for a.e. $t \in (\tau, T)$, $\forall p \in \partial J(y(t))$ $\frac{d}{dt}(J \circ y)(t) = (p, y'(t))_H$. In particular, for a.e. $t \in (\tau, T)$ $\frac{d}{dt}(J \circ y)(t) = (d(t), y'(t))_H$. Taking into account (2.125) we finally obtain the necessary statement.

The lemma is proved.

Lemma 2.20. *Let* $T > 0$. *Then any weak solution of Problem (2.120) on* $[0, T]$ *can be extended to a global one defined on* $[0, +\infty)$.

Proof. The statement of this lemma follows from Lemmas 2.17–2.19, Conditions (2.121), (2.122) and from the next estimates: $\forall \tau < T$, $\forall \varphi_\tau \in E$, $\forall \varphi(\cdot) = (y(\cdot), y'(\cdot))^T \in \mathscr{D}_{\tau,T}(\varphi_\tau)$, $\forall t \in [\tau, T]$ $2c_3 + \left(1 + \frac{2c_3}{\lambda_1}\right) \|y(\tau)\|_V^2 + \|y'(\tau)\|_H^2 \geq 2\mathscr{V}(\varphi(\tau)) \geq 2\mathscr{V}(\varphi(t)) = \|y(t)\|_V^2 + \|y'(t)\|_H^2 + 2J(y(t)) \geq \left(1 - \frac{\mu}{\lambda_1}\right) \|y(t)\|_V^2 + \|y'(t)\|_H^2 - 2c_4$.

The lemma is proved.

For an arbitrary $\varphi_0 \in E$ let $\mathscr{D}(\varphi_0)$ be the set of all weak solutions (defined on $[0, +\infty)$) of problem (2.120) with initial data $\varphi(0) = \varphi_0$. We remark that from the proof of Lemma 2.20 we obtain the next corollary.

Corollary 2.2. *For any* $\varphi_0 \in E$ *and* $\varphi \in \mathscr{D}(\varphi_0)$ *the next inequality is fulfilled:*

$$\|\varphi(t)\|_E^2 \leq \frac{\lambda_1 + 2c_3}{\lambda_1 - \mu} \|\varphi(0)\|_E^2 + \frac{2(c_3 + c_4)\lambda_1}{\lambda_1 - \mu} \quad \forall t > 0. \tag{2.126}$$

From Corollary 2.2 and Conditions $\underline{H(R)}$, $\underline{H(G)}$, $\underline{H(J)}$, $(\underline{H_0})$ in a standard way we obtain such proposition.

Theorem 2.5. *Let* $\tau < T$, $\{\varphi_n(\cdot)\}_{n\geq 1} \subset W_\tau^T$ *be an arbitrary sequence of weak solutions of (2.120) on* $[\tau, T]$ *such that* $\varphi_n(\tau) \to \varphi_\tau$ *weakly in* E, $n \to +\infty$, *and let* $\{t_n\}_{n\geq 1} \subset [\tau, T]$ *be a sequence such that* $t_n \to t_0$, $n \to +\infty$. *Then there exist* $\varphi \in \mathscr{D}_{\tau,T}(\varphi_\tau)$ *such that up to a subsequence* $\varphi_n(t_n) \to \varphi(t_0)$ *weakly in* E, $n \to +\infty$.

Proof. We prove this theorem in several steps.

Step 1. Let $\tau < T$, $\{\varphi_n(\cdot) = (u_n(\cdot), u_n'(\cdot))\}_{n\geq 1} \subset W_\tau^T$ be an arbitrary sequence of weak solutions of (2.120) on $[\tau, T]$ and $\{t_n\}_{n\geq 1} \subset [\tau, T]$:

$$\varphi_n(\tau) \to \varphi_\tau \text{ weakly in } E, \quad t_n \to t_0, \quad n \to +\infty. \tag{2.127}$$

In virtue of Corollary 2.2 we have that $\{\varphi_n(\cdot)\}_{n\geq 1}$ is bounded on $W_\tau^T \subset L_\infty(\tau, T; E)$. Therefore up to a subsequence $\{\varphi_{n_k}(\cdot)\}_{k\geq 1} \subset \{\varphi_n(\cdot)\}_{n\geq 1}$ we have:

$$
\begin{aligned}
&u_{n_k} \to u \text{ weakly star in } L_\infty(\tau,T;V),\\
&u'_{n_k} \to u' \text{ weakly star in } L_\infty(\tau,T;H),\\
&u''_{n_k} \to u'' \text{ weakly star in } L_\infty(\tau,T;V^*),\\
&d_{n_k} \to d \text{ weakly star in } L_\infty(\tau,T;H),\\
&u_{n_k} \to u \text{ in } L_2(\tau,T;H),\\
&u_{n_k}(t) \to u(t) \text{ in } H \text{ for a.e. } t\in[\tau,T],\\
&u'_{n_k}(t) \to u'(t) \text{ in } V^* \text{ for a.e. } t\in(\tau,T),\\
&Ru'_{n_k} \to Ru' \text{ weakly in } L_2(\tau,T;H),\\
&Gu_{n_k} \to Gu \text{ weakly in } L_2(\tau,T;V^*),\ k\to+\infty,
\end{aligned}
\tag{2.128}
$$

where $\forall n \geq 1\ d_n \in L_2(\tau,T;H)$,

$$
u''_n(t)+Ru'_n(t)+d_n(t)+Gu_n(t) = F,\ d_n(t)\in\partial j(u_n(t)) \text{ for a.e. } t\in(\tau,T).
\tag{2.129}
$$

Step 2. As ∂j is demiclosed is a standard way we get that $d(\cdot)\in\partial j(u(\cdot))$, $\varphi := (u,u')\in\mathscr{D}_{\tau,T}(\varphi_\tau)\subset W_\tau^T$.

Step 3. For a fixed $h\in V$ from (2.128) it follows that the sequence of real functions $(u_{n_k}(\cdot),h)$, $(u'_{n_k}(\cdot),h):[\tau,T]\to\mathbf{R}$ is uniformly bounded and equipotentionally continuous one. Taking into account (2.128), (2.126) and density of the embedding $V\subset H$ we obtain that $u'_{n_k}(t_{n_k})\to u'(t_0)$ weakly in H and $u_{n_k}(t_{n_k})\to u(t_0)$ weakly in V, $k\to+\infty$, whence it follows that the first part of this theorem is fulfilled.

The theorem is proved.

Theorem 2.6. *Let $\tau<T$, $\{\varphi_n(\cdot)\}_{n\geq1}\subset W_\tau^T$ be an arbitrary sequence of weak solutions of (2.120) on $[\tau,T]$ such that $\varphi_n(\tau)\to\varphi_\tau$ strongly in E, $n\to+\infty$, then up to a subsequence $\varphi_n(\cdot)\to\varphi(\cdot)$ in $C([\tau,T];E)$, $n\to+\infty$.*

Proof. Let $\tau<T$, $\{\varphi_n(\cdot)=(u_n(\cdot),u'_n(\cdot))^T\}_{n\geq1}\subset W_\tau^T$ be an arbitrary sequence of weak solutions of (2.120) on $[\tau,T]$ and $\{t_n\}_{n\geq1}\subset[\tau,T]$:

$$
\varphi_n(\tau)\to\varphi_\tau \text{ strongly in } E,\ n\to+\infty.
\tag{2.130}
$$

From Theorem 2.5 we have that there exist $\varphi\in\mathscr{D}_{\tau,T}(\varphi_\tau)$ such that up to a subsequence $\{\varphi_{n_k}(\cdot)\}_{k\geq1}\subset\{\varphi_n(\cdot)\}_{n\geq1}$, $\varphi_{n_k}(\cdot)\to\varphi(\cdot)$ weakly in E uniformly on $[\tau,T]$, $k\to+\infty$. Let us prove that

$$
\varphi_{n_k}\to\varphi \text{ in } W_\tau^T,\quad k\to+\infty.
\tag{2.131}
$$

By contradiction suppose the existence of $L>0$ and subsequence $\{\varphi_{k_j}\}_{j\geq1}\subset\{\varphi_{n_k}\}_{k\geq1}$ such that $\forall j\geq1\ \max\limits_{t\in[\tau,T]}\|\varphi_{k_j}(t)-\varphi(t)\|_E=\|\varphi_{k_j}(t_j)-\varphi(t_j)\|_E\geq L$. Without loss of generality we suggest that $t_j\to t_0\in[\tau,T]$, $j\to+\infty$. Therefore, by virtue of the continuity of $\varphi:[\tau,T]\to E$, we have

$$
\varliminf_{j\to+\infty}\|\varphi_{k_j}(t_j)-\varphi(t_0)\|_E\geq L.
\tag{2.132}
$$

On the other hand we prove that

$$\varphi_{k_j}(t_j) \to \varphi(t_0) \text{ in } E, \;\; j \to +\infty. \tag{2.133}$$

Firstly we remark that

$$\varphi_{k_j}(t_j) \to \varphi(t_0) \text{ weakly in } E, \quad j \to +\infty \tag{2.134}$$

(see Theorem 2.5 for details). Secondly let us prove that

$$\overline{\lim_{j\to+\infty}} \|\varphi_{k_j}(t_j)\|_E \le \|\varphi(t_0)\|_E. \tag{2.135}$$

Since J is sequentially weakly continuous, $\mathscr{V}$ is sequentially weakly lower semicontinuous on E. Hence, we obtain

$$\mathscr{V}(\varphi(t_0)) \le \underline{\lim_{j\to+\infty}} \mathscr{V}(\varphi_{k_j}(t_j)), \quad \int_\tau^{t_0} \|u'(s)\|_H^2 ds \le \underline{\lim_{j\to+\infty}} \int_\tau^{t_j} \|u'_{k_j}(s)\|_H^2 ds, \tag{2.136}$$

and hence

$$\mathscr{V}(\varphi(t_0)) + \gamma \int_\tau^{t_0} \|u'(s)\|_H^2 ds \le \underline{\lim_{j\to+\infty}} \left(\mathscr{V}(\varphi_{k_j}(t_j)) + \gamma \int_\tau^{t_j} \|u'_{k_j}(s)\|_H^2 ds \right). \tag{2.137}$$

Since by the energy equation both sides of (2.137) equal $\mathscr{V}(\varphi(\tau))$ (see Lemma 2.19), it follows from (2.136) that $\mathscr{V}(\varphi_{k_j}(t_j)) \to \mathscr{V}(\varphi(t_0))$, $j \to +\infty$ and (2.135). Convergence (2.133) directly follows from (2.134), (2.135) and [16, Chap. I]. To finish the proof of the theorem we remark that (2.133) contradicts (2.132). Therefore, (2.131) is true.

The theorem is proved.

We define the m-semiflow $\mathscr{G}$ as $\mathscr{G}(t, \xi_0) = \{\xi(t) \mid \xi(\cdot) \in \mathscr{D}(\xi_0)\}$, $t \ge 0$. Denote the set of all nonempty (nonempty bounded) subsets of E by $P(E)$ ($\beta(E)$). We remark that the multivalued map $\mathscr{G} : \mathbf{R}_+ \times E \to P(E)$ is *strict m-semiflow*, i.e. (see Lemma 2.18) $\mathscr{G}(0, \cdot) = \mathrm{Id}$ (the identity map), $\mathscr{G}(t+s, x) = \mathscr{G}(t, \mathscr{G}(s, x))$ $\forall x \in E, t, s \in \mathbf{R}_+$. Further $\varphi \in \mathscr{G}$ will mean that $\varphi \in \mathscr{D}(\xi_0)$ for some $\xi_0 \in E$.

Definition 2.2. The m-semiflow $\mathscr{G}$ is called *asymptotically compact*, if for any sequence $\varphi_j \in \mathscr{G}$ with $\varphi_j(0)$ bounded, and for any sequence $t_j \to +\infty$, the sequence $\varphi_j(t_j)$ has a convergent subsequence.

Theorem 2.7. *The m-semiflow $\mathscr{G}$ is asymptotically compact.*

Proof. Let $\xi_n \in \mathscr{G}(t_n, v_n)$, $v_n \in B \in \beta(E)$, $n \ge 1$, $t_n \to +\infty$, $n \to +\infty$. Let us check the precompactness of $\{\xi_n\}_{n\ge1}$ in E. In order to do that without loss of the generality it is sufficiently to extract a convergent in E subsequence from $\{\xi_n\}_{n\ge1}$. From Corollary 2.2 we obtain that there exist such $\{\xi_{n_k}\}_{k\ge1}$ and $\xi \in E$ that $\xi_{n_k} \to \xi$ weakly in E, $\|\xi_{n_k}\|_E \to a \ge \|\xi\|_E$, $k \to +\infty$. Show that $a \le \|\xi\|_E$. Let us fix an arbitrary $T_0 > 0$. Then for rather big $k \ge 1$ $\mathscr{G}(t_{n_k}, v_{n_k}) \subset \mathscr{G}(T_0, \mathscr{G}(t_{n_k} - T_0, v_{n_k}))$. Hence $\xi_{n_k} \in \mathscr{G}(T_0, \beta_{n_k})$, where $\beta_{n_k} \in \mathscr{G}(t_{n_k} - T_0, v_{n_k})$ and $\sup\limits_{k\ge1} \|\beta_{n_k}\|_E < +\infty$ (see Corollary 2.2). From Theorem 2.5 for some $\{\xi_{k_j}, \beta_{k_j}\}_{j\ge1} \subset \{\xi_{n_k}, \beta_{n_k}\}_{k\ge1}$, $\beta_{T_0} \in E$ we obtain:

$$\xi \in \mathscr{G}(T_0, \beta_{T_0}), \quad \beta_{k_j} \to \beta_{T_0} \text{ weakly in } E, \quad j \to +\infty. \tag{2.138}$$

From the definition of $\mathscr{G}$ we set: $\forall j \ge 1$ $\xi_{k_j} = (y_j(T_0), y_j'(T_0))^T$, $\beta_{k_j} = (y_j(0), y_j'(0))^T$, $\xi = (y_0(T_0), y_0'(T_0))^T$, $\beta_{T_0} = (y_0(0), y_0'(0))^T$, where $\varphi_j = (y_j, y_j')^T \in C([0, T_0]; E)$, $y_j'' \in L_2(0, T_0; V^*)$, $d_j \in L_\infty(0, T_0; H)$,

$$y_j''(t) + Ry_j'(t) + Gy_j(t) + d_j(t) = \bar{0}, \quad d_j(t) \in \partial J(y_j(t)) \quad \text{for a.e.} \quad t \in (0, T_0).$$

Let for each $t \in [0, T_0]$ $I(\varphi_j(t)) := \frac{1}{2}\|\varphi_j(t)\|_E^2 + J(y_j(t)) + \frac{\gamma}{2}(y_j'(t), y_j(t))$. Then, in virtue of Lemma 2.16, [33, Lemma 4.1,p. 78] and [33, Lemma 3.1,p. 71], $\frac{dI(\varphi_j(t))}{dt} = -\gamma I(\varphi_j(t)) + \gamma \mathscr{H}(\varphi_j(t))$, for a.e. $t \in (0, T_0)$, where $\mathscr{H}(\varphi_j(t)) = J(y_j(t)) - \frac{1}{2}(d_j(t), y_j(t))$.

From (2.126), (2.138) we have $\exists \bar{R} > 0 : \forall j \ge 0\ \forall t \in [0, T_0]$ $\|y_j'(t)\|_H^2 + \|y_j(t)\|_V^2 \le \bar{R}^2$. Moreover,

$$\begin{aligned} &y_j \to y_0 \text{ weakly in } L_2(0, T_0; V), \quad y_j' \to y_0' \text{ weakly in } L_2(0, T_0; H),\\ &y_j \to y_0 \text{ in } L_2(0, T_0; H), \qquad d_j \to d \text{ weakly in } L_2(0, T_0; H),\\ &y_j'' \to y_0'' \text{ weakly in } L_2(0, T_0; V^*),\ \forall t \in [0, T_0] \quad y_j(t) \to y_0(t) \text{ in } H,\ j \to +\infty. \end{aligned} \tag{2.139}$$

For any $j \ge 0$ and $t \in [0, T_0]$ $I(\varphi_j(t)) = I(\varphi_j(0))e^{-\gamma t} + \int\limits_0^t \mathscr{H}(\varphi_j(s))e^{-\gamma(t-s)}ds$, in particular $I(\varphi_j(T_0)) = I(\varphi_j(0))e^{-\gamma T_0} + \int\limits_0^{T_0} \mathscr{H}(\varphi_j(s))e^{-\gamma(T_0-s)}ds$. From (2.139) and Lemma 2.16 we have $\int\limits_0^{T_0} \mathscr{H}(\varphi_j(s))e^{-\gamma(T_0-s)}ds \to \int\limits_0^{T_0} \mathscr{H}(\varphi_0(s))e^{-\gamma(T_0-s)}ds$, $j \to +\infty$. Therefore, $\overline{\lim\limits_{j\to+\infty}} I(\varphi_j(T_0)) \le \overline{\lim\limits_{j\to+\infty}} I(\varphi_j(0))e^{-\gamma T_0} + \int\limits_0^{T_0} \mathscr{H}(\varphi_0(s))e^{-\gamma(T_0-s)} ds = I(\varphi_0(T_0)) + \left[\overline{\lim\limits_{j\to+\infty}} I(\varphi_j(0)) - I(\varphi_0(0))\right] e^{-\gamma T_0} \le I(\varphi_0(T_0)) + \bar{\bar{c}} e^{-\gamma T_0}$, where $\bar{\bar{c}}$ does not depend on $T_0 > 0$. On the other hand, from (2.139) we have $\overline{\lim\limits_{j\to+\infty}} I(\varphi_j(T_0)) \ge \frac{1}{2} \lim\limits_{j\to+\infty} \|\varphi_j(T_0)\|_E^2 + J(y_0(T_0)) + \frac{\gamma}{2}(y_0'(T_0), y_0(T_0))$. Therefore we obtain: $\frac{1}{2}a^2 \le \frac{1}{2}\|\xi\|_E^2 + \bar{\bar{c}} e^{-\gamma T_0}$ $\forall T_0 > 0$. Thus, $a \le \|\xi\|_E$.

The Theorem is proved.

Let us consider the family $\mathscr{K}_+ = \cup_{y_0 \in E} \mathscr{D}(y_0)$ of all weak solutions of the inclusion (2.120), defined on $[0, +\infty)$. Note that $\mathscr{K}_+$ is *translation invariant one*, i.e. $\forall u(\cdot) \in \mathscr{K}_+$, $\forall h \geq 0$ $u_h(\cdot) \in \mathscr{K}_+$, where $u_h(s) = u(h+s)$, $s \geq 0$. On $\mathscr{K}_+$ we set the *translation semigroup* $\{T(h)\}_{h \geq 0}$, $T(h)u(\cdot) = u_h(\cdot)$, $h \geq 0$, $u \in \mathscr{K}_+$. In view of the translation invariance of $\mathscr{K}_+$ we conclude that $T(h)\mathscr{K}_+ \subset \mathscr{K}_+$ as $h \geq 0$.

On $\mathscr{K}_+$ we consider a topology induced from the Fréchet space $C^{loc}(\mathbf{R}_+; E)$. Note that

$$f_n(\cdot) \to f(\cdot) \text{ in } C^{loc}(\mathbf{R}_+; E) \iff \forall M > 0\ \Pi_M f_n(\cdot) \to \Pi_M f(\cdot) \text{ in } C([0, M]; E),$$

where Π_M is the restriction operator to the interval $[0, M]$ [37, p. 179]. We denote the restriction operator to $[0, +\infty)$ by Π_+.

Let us consider the autonomous inclusion (2.120) on the entire time axis. Similarly to the space $C^{loc}(\mathbf{R}_+; E)$ the space $C^{loc}(\mathbf{R}; E)$ is endowed with the topology of local uniform convergence on each interval $[-M, M] \subset \mathbf{R}$ (cf. [37, p. 180]). A function $u \in C^{loc}(\mathbf{R}; E) \cap L_\infty(\mathbf{R}; E)$ is said to be *a complete trajectory* of the inclusion (2.120), if $\forall h \in \mathbf{R}$ $\Pi_+ u_h(\cdot) \in \mathscr{K}_+$ [37, p. 180]. Let $\mathscr{K}$ be a family of *all complete trajectories* of the inclusion (2.120). Note that $\forall h \in \mathbf{R}$, $\forall u(\cdot) \in \mathscr{K}$ $u_h(\cdot) \in \mathscr{K}$. We say that the complete trajectory $\varphi \in \mathscr{K}$ is *stationary* if $\varphi(t) = z$ for all $t \in \mathbf{R}$ for some $z \in E$. Following [4, p.486] we denote the set of rest points of $\mathscr{G}$ by $Z(\mathscr{G})$. We remark that $Z(\mathscr{G}) = \{(\bar{0}, u) \mid u \in V,\ G(u) + \partial J(u) \ni \bar{0}\}$.

From Conditions $H(G)$ and $H(J)$ it follows that

Lemma 2.21. *The set $Z(\mathscr{G})$ is bounded in E.*

From Lemma 2.19 the existence of Lyapunov function (see [4, p. 486]) for $\mathscr{G}$ is follows.

Lemma 2.22. *A functional $\mathscr{V} : E \to \mathbf{R}$, defined by (2.124), is a Lyapunov function for $\mathscr{G}$.*

2.7 Asymptotic Behavior of the Second-Order Evolution Inclusions

Here, we consider at first long-time behavior for state functions of viscoelastic fields that can be described with the second-order evolution inclusion. We can obtain the similar results for piezoelectric fields, analyzing the respective proofs. Therefore, we have a chain of results for global and trajectory attractors, presented in this section.

At first, we remark that the existence of the global attractor for Second-Order Evolution Inclusions and Hemivariational Inequalities considered in Sect. 2.6 (piezoelectric fields) directly follows from Lemmas 2.17, 2.18, 2.21, and 2.22; Theorems 2.5–2.7; and [5, Theorem 2.7].

Theorem 2.8. *The m-semiflow $\mathscr{G}$ has the invariant compact in the phase space E global attractor $\mathscr{A}$. For each $\psi \in \mathscr{K}$, the limit sets*

$$\alpha(\psi) = \{z \in E \mid \psi(t_j) \to z \text{ for some sequence } t_j \to -\infty\},$$
$$\omega(\psi) = \{z \in E \mid \psi(t_j) \to z \text{ for some sequence } t_j \to +\infty\}$$

are connected subsets of $Z(G)$ on which $\mathscr{V}$ is constant. If $Z(G)$ is totally disconnected (in particular, if $Z(G)$ is countable), the limits

$$z_- = \lim_{t\to-\infty} \psi(t), \quad z_+ = \lim_{t\to+\infty} \psi(t)$$

exist and z_-, z_+ are rest points; furthermore, $\varphi(t)$ tends to a rest point as $t \to +\infty$ for every $\varphi \in \mathscr{K}_+$.

2.7.1 Existence of the Global Attractor

First, we consider constructions presented in [4, 24]. Denote the set of all nonempty (nonempty bounded) subsets of E by $P(E)$ ($\beta(E)$). We recall that the multivalued map $G : \mathbf{R} \times E \to P(E)$ is said to be a *m-semiflow* if:

(a) $G(0, \cdot) = \mathrm{Id}$ (the identity map).

(b) $G(t + s, x) \subset G(t, G(s, x))\ \forall x \in E, t, s \in \mathbf{R}_+$;
m-semiflow is a *strict* one if $G(t + s, x) = G(t, G(s, x))\ \forall x \in E, t, s \in \mathbf{R}_+$.

From Lemmas 2.13 and 2.15, it follows that any weak solution can be extended to a global one defined on $[0, +\infty)$. For an arbitrary $\xi_0 = (a, b)^T \in E$, let $\mathscr{D}(\xi_0)$ consists of pairs of functions $(u(\cdot), u'(\cdot))^T$, defined on $[0, +\infty)$, where $(u(\cdot), u'(\cdot))^T$ is a weak solution (defined on $[0, +\infty)$) of the problem (2.78) with initial data $u(0) = a$, $u'(0) = b$.

We define the semiflow G as $G(t, \xi_0) = \{\xi(t) \mid \xi(\cdot) \in \mathscr{D}(\xi_0)\}$.

Lemma 2.23. *G is the strict m-semiflow.*

Proof. Let $\xi \in G(t + s, \xi_0)$. Then $\xi = \psi(t + s)$, where $\psi(\cdot) \in \mathscr{D}(\xi_0)$. From Lemma 2.14, it follows that $\varphi(\cdot) = \psi(s + \cdot) \in \mathscr{D}(\psi(s))$. Hence, $\xi = \varphi(t) \in G(t, \psi(s)) \subset G(t, G(s, \xi_0))$.

Vice versa, if $\xi \in G(t, G(s, \xi_0))$, then $\exists \psi(\cdot) \in \mathscr{D}(\xi_0), \varphi(\cdot) \in \mathscr{D}(\psi(s))$: $\xi = \varphi(t)$. Define the map

$$\phi(\zeta) = \begin{cases} \psi(\zeta), & \zeta \in [0, s], \\ \varphi(\zeta - s), & \zeta \in [s, t + s]. \end{cases}$$

From Lemma 2.14, it follows that $\phi(\cdot) \in \mathscr{D}(\xi_0)$. Hence, $\xi = \phi(t+s) \in G(t+s, \xi_0)$.

We recall that the set $\mathscr{A}$ is said to be a *global attractor G*, if

1. $\mathscr{A}$ is negatively semiinvariant (i.e., $\mathscr{A} \subset G(t, \mathscr{A})\ \forall t \geq 0$).
2. $\mathscr{A}$ is attracting set, that is,

$$\mathrm{dist}(G(t, B), \mathscr{A}) \to 0, \quad t \to +\infty \quad \forall B \in \beta(E), \tag{2.140}$$

   where $\mathrm{dist}(C, D) = \sup_{c \in C} \inf_{d \in D} \|c - d\|_E$ is the Hausdorff semidistance.
3. For any closed set $Y \subset H$, satisfying (2.140), we have $\mathscr{A} \subset Y$ (minimality).

The global attractor is said to be *invariant*, if $\mathscr{A} = G(t, \mathscr{A})\ \forall t \geq 0$.
Note that from the definition of the global attractor, it follows that it is unique.
We prove the existence of the invariant compact global attractor.

Theorem 2.9. *The m-semiflow G has the invariant compact in the phase space E global attractor $\mathscr{A}$.*

Proof. From Lemma 2.15, it follows that $\exists R, \bar{\alpha}, \bar{\beta} > 0$:

$$\forall \xi_0 \in E,\ \forall \xi(\cdot) \in \mathscr{D}(\xi_0),\ \forall t \geq 0 \quad \|\xi(t)\|_E^2 \leq \bar{\beta} \|\xi_0\|_E^2 e^{-\tilde{\alpha} t} + \frac{R^2}{2}. \tag{2.141}$$

Thus, the ball $B_0 = \{u \in E \mid \|u\|_E \leq R\}$ is the absorbing set, that is, $\forall B \in \beta(E)$ $\exists T(B) > 0$: $\forall t \geq T(B)$ $G(t, B) \subset B_0$. In particular, from (2.141), it follows that the set $\cup_{t \geq 0} G(t, B)$ is bounded one in E $\forall B \in \beta(E)$. Note also that from Theorem 2.4, it follows that the map $G(t, \cdot) : E \to \beta(E)$ takes compact values.

The upper semicontinuity of the map $u_0 \to G(t, u_0)$ (cf. [2, Definition 1.4.1, p. 38]) follows from that fact that the given map is compact-valued one and Theorem 2.4 (cf. [18]). In order to do that, it is sufficient to show [3, p. 45] that $\forall \xi_0 \in E$, $\forall \varepsilon > 0$ $\exists \delta(\xi_0, \varepsilon) > 0$: $\forall \xi \in B_\delta(\xi_0)$ $G(t, \xi) \subset B_\varepsilon(G(t, \xi_0)) = \{z \in E \mid \mathrm{dist}(z, G(t, \xi_0)) < \varepsilon\}$. If it is not true, then there exist $\xi_0 \in E$, $\varepsilon > 0$, $\{\delta_n\}_{n \geq 1} \subset (0, +\infty)$, $\{\xi_n\}_{n \geq 1} \subset E$ such that $\forall n \geq 1$ $\xi_n \in B_{\delta_n}(\xi_0)$, $G(t, \xi_n) \not\subset B_\varepsilon(G(t, \xi_0))$ and $\delta_n \to 0$, $n \to +\infty$. Then $\forall n \geq 1$ $\exists \zeta_n(\cdot) \in \mathscr{D}(\xi_n)$: $\zeta_n(t) \notin B_\varepsilon(G(t, \xi_0))$. Since $\xi_n \to \xi_0$ in E, $n \to +\infty$, then from Theorem 2.4, it follows that $\zeta_n(t) \to \zeta(t) \in G(t, \xi_0)$ in E, $n \to +\infty$, for some $\zeta(\cdot) \in \mathscr{D}(\xi_0)$. We obtain a contradiction with $\forall n \geq 1$ $\|\zeta_n(t) - \zeta(t)\|_E \geq \varepsilon$.

Now, we check the upper asymptotic semicompactness of the m-semiflow G. Let $\xi_n \in G(t_n, v_n)$, $v_n \in B \in \beta(E)$, $n \geq 1$, $t_n \to +\infty$, $n \to +\infty$. Let us check the precompactness of $\{\xi_n\}_{n \geq 1}$ in E. In order to do that without loss of the generality, it is sufficiently to extract a convergent in E subsequence from $\{\xi_n\}_{n \geq 1}$.

From Lemma 2.15 and Theorem 2.4, we obtain that there exist such $\{\xi_{n_k}\}_{k \geq 1}$ and $\xi \in E$ that

$$\xi_{n_k} \to \xi \text{ weakly in } E, \quad \|\xi_{n_k}\|_E \to a \geq \|\xi\|_E, \quad k \to +\infty. \tag{2.142}$$

Show that

$$a \leq \|\xi\|_E. \tag{2.143}$$

Let us fix an arbitrary $T_0 > \sqrt{\lambda_1}$, where $\lambda_1 > 0$ is the constant from (2.86). Then for rather big $k \geq 1$ $G(t_{n_k}, v_{n_k}) \subset G(T_0, G(t_{n_k} - T_0, v_{n_k})) \subset G(T_0, B_0)$. Hence, $\xi_{n_k} \in G(T_0, \beta_{n_k})$, where $\beta_{n_k} \in G(t_{n_k} - T_0, v_{n_k})$ and $\|\beta_{k_j}\|_E \leq R\ \forall j \geq 1$. From Lemma 2.15, Theorem 2.4, and (2.142) for some $\{\xi_{k_j}, \beta_{k_j}\}_{j\geq 1} \subset \{\xi_{n_k}, \beta_{n_k}\}_{k\geq 1}$, $\beta_{T_0} \in E$, we obtain: $\forall j \geq 1$

$$\xi \in G(T_0, \beta_{T_0}), \quad \beta_{k_j} \to \beta_{T_0} \text{ weakly in } E, \quad j \to +\infty. \tag{2.144}$$

From the definition of G, we obtain that $\forall j \geq 1$

$$\xi_{k_j} = \begin{pmatrix} y_j(T_0) \\ y_j'(T_0) \end{pmatrix}, \ \beta_{k_j} = \begin{pmatrix} y_j(0) \\ y_j'(0) \end{pmatrix}, \quad \xi = \begin{pmatrix} y_0(T_0) \\ y_0'(T_0) \end{pmatrix}, \ \beta_{T_0} = \begin{pmatrix} y_0(0) \\ y_0'(0) \end{pmatrix},$$

where $y_j \in C([0, T_0]; V)$: $y_j' \in W_{0,T_0}$ and

$$y_j'' + d_j + B_{0,T_0} y_j = \bar{0}, \quad d_j \in A_{0,T_0}(y_j) - f_{0,T_0}, \ j \geq 0. \tag{2.145}$$

Let us fix an arbitrary $\varepsilon \in (0, \sqrt{\lambda_1})$. From (2.141), we have:

$$\forall j \geq 0, \ \forall t \in [0, T_0] \quad \|y_j'(t)\|_H^2 + \|y_j(t)\|_V^2 \leq R^2(\bar{\beta} + 1/2) =: \bar{R}^2. \tag{2.146}$$

From the proof of Theorem 2.4, we obtain that

$$\begin{aligned} &\exists C > 0: \quad \|y_j'\|_{X_{0,T_0}} + \|y_j''\|_{X^*_{0,T_0}} + \|d_j\|_{X^*_{0,T_0}} \leq C \quad \forall j \geq 0; \\ &y_j' \to y_0' \text{ in } C([\varepsilon, T_0]; H), \\ &y_j' \to y_0' \text{ weakly in } W_{0,T_0}, \\ &y_j' \to y_0' \text{ in } L_2(0, T_0; H), \\ &d_j \to d_0 \text{ weakly in } X^*_{0,T_0}, \\ &v_j \to v_0 \text{ in } C([0, T_0]; V), \quad j \to +\infty, \\ &\forall j \geq 0, \ \forall t \in [0, T_0] \quad v_j(t) = y_j(t) - y_j(0). \end{aligned} \tag{2.147}$$

Let us consider the next denotations:

$$Y_j(t) = \frac{1}{2}\left[\|y_j(t)\|_V^2 + \|y_j'(t)\|_H^2\right] + \varepsilon(y_j'(t), y_j(t)), \quad t \in [0, T_0], \ j \geq 0.$$

Then $\forall j \geq 0$ and for a.e. $t \in (0, T_0)$

$$\begin{aligned} \frac{dY_j(t)}{dt} &= -2\varepsilon Y_j(t) + 2\varepsilon\|y_j'(t)\|_H^2 - \langle d_j(t), y_j'(t)\rangle_V - \varepsilon\langle d_j(t), y_j(t)\rangle_V \\ &\quad + 2\varepsilon^2(y_j'(t), y_j(t)). \end{aligned}$$

Hence,

$$\frac{d}{dt}\left[Y_j(t)e^{2\varepsilon t}\right] = 2\varepsilon\|y_j'(t)\|_H^2 e^{2\varepsilon t} - \langle d_j(t), y_j'(t)\rangle_V e^{2\varepsilon t}$$
$$-\varepsilon\langle d_j(t), y_j(t)\rangle_V e^{2\varepsilon t}$$
$$+2\varepsilon^2(y_j'(t), y_j(t))^{2\varepsilon t}.$$

Thus, $\forall j \geq 0$

$$\begin{aligned} Y_j(T_0) = Y_j(0)e^{-2\varepsilon T_0} &+ 2\varepsilon\int_0^{T_0} \|y_j'(t)\|_H^2 e^{-2\varepsilon(T_0-t)}dt \\ &- \int_0^{T_0} \langle d_j(t), y_j'(t)\rangle_V e^{-2\varepsilon(T_0-t)}dt \\ &-\varepsilon\int_0^{T_0} \langle d_j(t), y_j(t)\rangle_V e^{-2\varepsilon(T_0-t)}dt \\ &+2\varepsilon^2\int_0^{T_0} (y_j'(t), y_j(t))e^{-2\varepsilon(T_0-t)}dt. \end{aligned} \tag{2.148}$$

From (2.147), for every $j \geq 1$ and for a.e. $t \in (0, T_0)$, we obtain:

$$2\varepsilon\int_0^{T_0} \|y_j'(t)\|_H^2 e^{-2\varepsilon(T_0-t)}dt \to 2\varepsilon\int_0^{T_0} \|y_0'(t)\|_H^2 e^{-2\varepsilon(T_0-t)}dt, \quad j \to +\infty. \tag{2.149}$$

In view of (2.145), $\forall j \geq 0$ and a.e. $t \in (0, T_0)$

$$\langle d_j(t), y_j'(t)\rangle_V = -\frac{1}{2}\frac{d}{dt}\left[\|y_j(t)\|_V^2 + \|y_j'(t)\|_H^2\right]$$
$$= -\frac{1}{2}\frac{d}{dt}\left[\|v_j(t)\|_V^2 + 2\langle B_0 y_j(0), v_j(t)\rangle_V + \|y_j'(t)\|_H^2\right].$$

Taking into account (2.147), we have:

$$\lim_{j\to+\infty}\int_\varepsilon^{T_0} \langle d_j(t), y_j'(t)\rangle_V dt = \int_\varepsilon^{T_0} \langle d_0(t), y_0'(t)\rangle_V dt.$$

Further, following [20, pp. 7–10], from Sect. 2.2 and (2.147), we obtain that

$$\lim_{j\to+\infty}\int_{\varepsilon}^{T_0}|\langle d_j(t), y_j'(t)-y_0'(t)\rangle_V|dt=0,$$

and due to (2.147),

$$\int_{\varepsilon}^{T_0}\langle d_j(t), y_j'(t)\rangle_V e^{-2\varepsilon(T_0-t)}dt \to \int_{\varepsilon}^{T_0}\langle d_0(t), y_0'(t)\rangle_V e^{-2\varepsilon(T_0-t)}dt, \quad j\to+\infty. \tag{2.150}$$

From Condition (A_1) and (2.146), we obtain:

$$\forall j\geq 0 \quad \left|\int_0^{\varepsilon}\langle d_j(t), y_j'(t)\rangle_V e^{-2\varepsilon(T_0-t)}dt\right| \leq c(1+\bar{R})\bar{R}e^{-2\varepsilon(T_0-\varepsilon)}\varepsilon. \tag{2.151}$$

From Condition (A_3), we have that

$$\forall j\geq 0\,\exists z_j\in L_2(0,T_0;Z^*):\quad d_j(\cdot)=A_1y_j'(\cdot)+z_j(\cdot).$$

Taking into account Condition (A_3), (2.149), and [41, 42], we obtain that $y_j\to y_0$ in $L_2(0,T_0;Z)$, $z_j\to z_0$ weakly in $L_2(0,T_0;Z^*)$, $j\to+\infty$. Therefore,

$$\int_0^{T_0}\langle z_j(t), y_j(t)\rangle_Z e^{-2\varepsilon(T_0-t)}dt \to \int_0^{T_0}\langle z_0(t), y_0(t)\rangle_Z e^{-2\varepsilon(T_0-t)}dt,\ j\to+\infty,$$

$$\begin{aligned}
-\int_0^{T_0}\langle A_1y_j'(t), y_j(t)\rangle_V e^{-2\varepsilon(T_0-t)}dt &= -\int_0^{T_0} e^{-2\varepsilon(T_0-t)}\frac{d}{dt}\langle A_1y_j(t), y_j(t)\rangle_V dt\\
&= -\langle A_1y_j(T_0), y_j(T_0)\rangle_V+\langle A_1y_j(0), y_j(0)\rangle_V e^{-2\varepsilon T_0}\\
&\quad +2\varepsilon\int_0^{T_0}\langle A_1y_j(t), y_j(t)\rangle_V e^{-2\varepsilon(T_0-t)}dt\\
&\quad \leq \|A_1\|_{L(V;V^*)}(R^2e^{-2\varepsilon T_0}+\bar{R}^2).
\end{aligned}$$

Thus,

$$\overline{\lim_{j\to+\infty}}\left(-2\varepsilon\int_0^{T_0}\langle d_j(t), y_j(t)\rangle_V e^{-\varepsilon(T_0-t)}dt\right)$$
$$\le -\varepsilon\int_0^{T_0}\langle d_0(t), y_0(t)\rangle_V e^{-2\varepsilon(T_0-t)}dt + \|A_1\|_{L(V;V^*)}(R^2e^{-2\varepsilon T_0} + 2\bar{R}^2)\varepsilon. \tag{2.152}$$

In virtue of (2.90) and (2.146),

$$2\varepsilon^2\left|\int_0^{T_0}(y_j'(t), y_j(t))e^{-2\varepsilon(T_0-t)}dt\right| \le \frac{\varepsilon}{2\sqrt{\lambda_1}}\bar{R}^2. \tag{2.153}$$

Finally, from (2.90) and (2.148)–(2.152), we obtain:

$$\overline{\lim_{j\to+\infty}} Y_j(T_0) \le R^2\left(1+\frac{\varepsilon}{\sqrt{\lambda_1}}\right)e^{-2\varepsilon T_0}$$
$$+ Y_0(T_0) + \|A_1\|_{L(V;V^*)}(R^2e^{-2\varepsilon T_0} + 2\bar{R}^2)\varepsilon$$
$$+ 2c(1+\bar{R})\bar{R}e^{-2\varepsilon(T_0-\varepsilon)}\varepsilon + \frac{\varepsilon}{\sqrt{\lambda_1}}\bar{R}^2.$$

Thus, $\forall\varepsilon \in (0, \sqrt{\lambda_1})$, $\forall T_0 > \sqrt{\lambda_1}$

$$\frac{1}{2}a^2\left(1-\frac{\varepsilon}{\sqrt{\lambda_1}}\right) \le R^2\left(1+\frac{\varepsilon}{\sqrt{\lambda_1}}\right)e^{-2\varepsilon T_0} + \frac{1}{2}\|\xi\|_E^2\left(1+\frac{\varepsilon}{\sqrt{\lambda_1}}\right)$$
$$+ \|A_1\|_{L(V;V^*)}(R^2e^{-2\varepsilon T_0} + 2\bar{R}^2)\varepsilon$$
$$+ 2c(1+\bar{R})\bar{R}e^{-2\varepsilon(T_0-\varepsilon)}\varepsilon + \frac{\varepsilon}{\sqrt{\lambda_1}}\bar{R}^2.$$

Rushing $T_0 \to +\infty$ in the last inequality, we obtain: $\forall\varepsilon \in (0, \sqrt{\lambda_1})$

$$a^2\left(1-\frac{\varepsilon}{\sqrt{\lambda_1}}\right) \le \|\xi\|_E^2\left(1+\frac{\varepsilon}{\sqrt{\lambda_1}}\right) + 4\|A_1\|_{L(V;V^*)}\bar{R}^2\varepsilon + \frac{2\varepsilon}{\sqrt{\lambda_1}}\bar{R}^2. \tag{2.154}$$

Passing to the limit (as $\varepsilon \to 0+$) in the inequality (2.154), we obtain (2.143). From (2.142) to (2.143), it follows that $\xi_{n_k} \to \xi$ in E, $k \to +\infty$.

Thus, the existence of the global attractor with required properties directly follows from results from Chap. 1.

2.7.2 Existence of the Trajectory Attractor

Let us consider the family $\mathscr{K}_+ = \cup_{y_0 \in E} \mathscr{D}(y_0)$ of all weak solutions of the inclusion (2.78), defined on $[0, +\infty)$. Note that $\mathscr{K}_+$ is *translation invariant one*, that is, $\forall u(\cdot) \in \mathscr{K}_+$, $\forall h \geq 0$ $u_h(\cdot) \in \mathscr{K}_+$, where $u_h(s) = u(h+s)$, $s \geq 0$. On $\mathscr{K}_+$, we set the *translation semigroup* $\{T(h)\}_{h\geq 0}$, $T(h)u(\cdot) = u_h(\cdot)$, $h \geq 0$, $u \in \mathscr{K}_+$. In view of the translation invariance of $\mathscr{K}_+$, we conclude that $T(h)\mathscr{K}_+ \subset \mathscr{K}_+$ as $h \geq 0$.

We shall construct the attractor of the translation semigroup $\{T(h)\}_{h\geq 0}$, acting on $\mathscr{K}_+$. On $\mathscr{K}_+$, we consider a topology induced from the Fréchet space $C^{loc}(\mathbf{R}_+; E)$. Note that

$$f_n(\cdot) \to f(\cdot) \text{ in } C^{loc}(\mathbf{R}_+; E) \iff \forall M > 0\ \Pi_M f_n(\cdot) \to \Pi_M f(\cdot) \text{ in } C([0, M]; E),$$

where Π_M is the restriction operator to the interval $[0, M]$ [37, p. 179]. We denote the restriction operator to $[0, +\infty)$ by Π_+.

We recall that the set $\mathscr{P} \subset C^{loc}(\mathbf{R}_+; E) \cap L_\infty(\mathbf{R}_+; E)$ is said to be *an attracting one* for the trajectory space $\mathscr{K}_+$ of the inclusion (8) in the topology of $C^{loc}(\mathbf{R}_+; E)$, if for any bounded (in $L_\infty(\mathbf{R}_+; E)$) set $\mathscr{B} \subset \mathscr{K}_+$ and an arbitrary number $M \geq 0$, the next relation

$$\mathrm{dist}_{C([0,M];E)}(\Pi_M T(t)\mathscr{B}, \Pi_M \mathscr{P}) \to 0, \quad t \to +\infty \tag{2.155}$$

holds true.

A set $\mathscr{U} \subset \mathscr{K}_+$ is said to be *trajectory attractor* in the trajectory space $\mathscr{K}_+$ with respect to the topology of $C^{loc}(\mathbf{R}_+; E)$ (cf. [37, Definition 1.2, p. 179]), if:

(i) $\mathscr{U}$ is a compact set in $C^{loc}(\mathbf{R}_+; E)$ and bounded one in $L_\infty(\mathbf{R}_+; E)$.
(ii) $\mathscr{U}$ is strictly invariant with respect to $\{T(h)\}_{h\geq 0}$, that is, $T(h)\mathscr{U} = \mathscr{U}$ $\forall h \geq 0$.
(iii) $\mathscr{U}$ is an attracting set in the trajectory space $\mathscr{K}_+$ in the topology $C^{loc}(\mathbf{R}_+; E)$.

Note that from the definition of the trajectory attractor, it follows that it is unique.

Let us consider the inclusion (8) on the entire time axis. Similarly to the space $C^{loc}(\mathbf{R}_+; E)$, the space $C^{loc}(\mathbf{R}; E)$ is endowed with the topology of local uniform convergence on each interval $[-M, M] \subset \mathbf{R}$ (cf. [37, p. 180]). A function $u \in C^{loc}(\mathbf{R}; E) \cap L_\infty(\mathbf{R}; E)$ is said to be *a complete trajectory* of the inclusion (8), if $\forall h \in \mathbf{R}$ $\Pi_+ u_h(\cdot) \in \mathscr{K}_+$ [37, p. 180]. Let $\mathscr{K}$ be a family of all complete trajectories of the inclusion (8). Note that

$$\forall h \in \mathbf{R},\ \forall u(\cdot) \in \mathscr{K} \quad u_h(\cdot) \in \mathscr{K}. \tag{2.156}$$

The existence of the trajectory attractor and its structure properties follow from such theorem.

Theorem 2.10. *Let $\mathscr{A}$ be a global attractor from Theorem 2.9. Then there exists the trajectory attractor $\mathscr{P} \subset \mathscr{K}_+$ in the space $\mathscr{K}_+$. At that, the next formula takes place*

$$\mathscr{P} = \Pi_+ \mathscr{K} = \Pi_+ \{y \in \mathscr{K} \mid y(t) \in \mathscr{A}\ \forall t \in \mathbf{R}\}, \tag{2.157}$$

Proof. The proof repeats the proof of corresponding statement from work [18] and, it is based on results of Theorems 2.4, 2.9.

First, we prove some auxiliary lemmas.

Lemma 2.24. *The set $\mathscr{K}$ is nonempty, compact in $C^{loc}(\mathbf{R}; E)$, and bounded one in $L_\infty(\mathbf{R}; E)$. Moreover,*

$$\forall \xi(\cdot) \in \mathscr{K}, \ \forall t \in \mathbf{R} \quad \xi(t) \in \mathscr{A}, \tag{2.158}$$

where $\mathscr{A}$ is the global attractor from Theorem 2.9.

Proof.

Step 1. Let us show that $\mathscr{K} \neq \emptyset$. Note that in view of conditions (B_1), (B_2), (H_1), it follows that $\exists v \in V$: $B_0(v) = f_0$. We set $u(t) = v \ \forall t \in \mathbf{R}$. Then $(u, u')^T \in \mathscr{K} \neq \emptyset$.

Step 2. Let us prove (2.158). For any $y \in \mathscr{K}$ $\exists d > 0$: $\|y(t)\|_E \le d \ \forall t \in \mathbf{R}$. We set $B = \cup_{t \in \mathbf{R}}\{y(t)\} \in \beta(E)$. Note that $\forall \tau \in \mathbf{R}$, $\forall t \in \mathbf{R}_+$ $y(\tau) = y_{\tau-t}(t) \in G(t, y_{\tau-t}(0)) \subset G(t, B)$. From Theorem 2.9 and from (2.140), it follows that $\forall \varepsilon > 0 \ \exists T > 0$: $\forall \tau \in \mathbf{R}$ $\operatorname{dist}(y(\tau), \mathscr{A}) \le \operatorname{dist}(G(T, B), \mathscr{A}) < \varepsilon$. Hence, taking into account the compactness of $\mathscr{A}$ in E, for any $u(\cdot) \in \mathscr{K}$, $\tau \in \mathbf{R}$, it follows that $u(\tau) \in \mathscr{A}$.

Step 3. The boundedness of $\mathscr{K}$ in $L_\infty(\mathbf{R}_+; E)$ follows from (2.158) and the boundedness of $\mathscr{A}$ in E.

Step 4. Let us check the compactness of $\mathscr{K}$ in $C^{loc}(\mathbf{R}; E)$. In order to do that, it is sufficient to check the precompactness and completeness.

Step 4.1. Let us check the precompactness of $\mathscr{K}$ in $C^{loc}(\mathbf{R}; E)$. If it is not true, then in view of (2.156), $\exists M > 0$: $\Pi_M \mathscr{K}$ is not precompact set in $C([0, M]; E)$. Hence, there exists a sequence $\{v_n\}_{n \ge 1} \subset \Pi_M \mathscr{K}$ that has not a convergent subsequence in $C([0, M]; E)$. On the other hand, $v_n = \Pi_M u_n$, where $u_n \in \mathscr{K}$, $v_n(0) = u_n(0) \in \mathscr{A}$, $n \ge 1$. Since $\mathscr{A}$ is compact set in E (see Theorem 2.9), then in view of Theorem 2.4, $\exists \{v_{n_k}\}_{k \ge 1} \subset \{v_n\}_{n \ge 1}$, $\exists \eta \in E$, $\exists v(\cdot) \in \mathscr{D}_{0,M}(\eta)$: $v_{n_k}(0) \to \eta$ in E, $v_{n_k} \to v$ in $C([0, T]; E)$, $k \to +\infty$. We obtained a contradiction.

Step 4.2. Let us check the completeness of $\mathscr{K}$ in $C^{loc}(\mathbf{R}; E)$. Let $\{v_n\}_{n \ge 1} \subset \mathscr{K}$, $v \in C^{loc}(\mathbf{R}; E)$: $v_n \to v$ in $C^{loc}(\mathbf{R}; E)$, $n \to +\infty$. From the boundedness of $\mathscr{K}$ in $L_\infty(\mathbf{R}; E)$, it follows that $v \in L_\infty(\mathbf{R}; E)$. From Theorem 2.9, we have that $\forall M > 0$ the restriction $v(\cdot)$ to the interval $[-M, M]$ belongs to $\mathscr{D}_{-M,M}(v(-T))$. Therefore, $v(\cdot)$ is the complete trajectory of the inclusion (8). Thus, $v \in \mathscr{K}$.

Lemma 2.25. *Let $\mathscr{A}$ be a global attractor from Theorem 2.9. Then*

$$\forall y_0 \in \mathscr{A} \quad \exists y(\cdot) \in \mathscr{K}: \quad y(0) = y_0. \tag{2.159}$$

Proof. Let $y_0 \in \mathscr{A}$, $u(\cdot) \in \mathscr{D}(y_0)$. From (2.85) and (2.140), we obtain that $\forall t \in \mathbf{R}_+$ $y(t) \in \mathscr{A}$. From Theorem 2.9, it follows that $G(1, \mathscr{A}) = \mathscr{A}$. Therefore,

$$\forall \eta \in \mathscr{A} \quad \exists \xi \in \mathscr{A}, \ \exists \varphi_\eta(\cdot) \in \mathscr{D}_{0,1}(\xi): \quad \varphi_\eta(1) = \eta.$$

For any $t \in \mathbf{R}$, we set

$$y(t) = \begin{cases} u(t), & t \in \mathbf{R}_+, \\ \varphi_{y(-k+1)}(t+k), & t \in [-k, -k+1),\ k \in \mathbf{N}. \end{cases}$$

Note that $y \in C^{loc}(\mathbf{R}; E)$, $y(t) \in \mathscr{A}$ $\forall t \in \mathbf{R}$ (hence, $y \in L_\infty(\mathbf{R}; E)$), and in view of Lemma 2.14, $y \in \mathscr{K}$. At that, $y(0) = y_0$.

Let us continue the proof of the theorem. From Lemma 2.24 and the continuity of the operator $\Pi_+ : C^{loc}(\mathbf{R}; E) \to C^{loc}(\mathbf{R}_+; E)$, it follows that the set $\Pi_+\mathscr{K}$ is nonempty, compact in $C^{loc}(\mathbf{R}_+; E)$ and bounded one in $L_\infty(\mathbf{R}_+; E)$. Moreover, the second equality in (2.157) holds true. The strict invariance of $\Pi_+\mathscr{K}$ follows from the autonomy of the inclusion (8).

Let us prove that $\Pi_+\mathscr{K}$ is the attracting set for the trajectory space $\mathscr{K}_+$ in the topology of $C^{loc}(\mathbf{R}_+; E)$. Let $B \subset \mathscr{K}_+$ be a bounded set in $L_\infty(\mathbf{R}_+; E)$, $M \geq 0$. Let us check (2.155). If it is not true, then there exist sequences $t_n \to +\infty$, $v_n(\cdot) \in B$ such that

$$\forall n \geq 1 \quad \mathrm{dist}_{C([0,T];E)}(\Pi_M v_n(t_n + \cdot), \Pi_M \mathscr{K}) \geq \varepsilon. \tag{2.160}$$

On the other hand, from the boundedness of B in $L_\infty(\mathbf{R}_+; E)$, it follows that $\exists R > 0$: $\forall v(\cdot) \in B$, $\forall t \in \mathbf{R}_+$ $\|v(t)\|_E \leq R$. Thus, $\exists N \geq 1$: $\forall n \geq N$ $v_n(t_n) \in G(t_n, v_n(0)) \subset G(1, G(t_n - 1, v_n(0))) \subset G(1, \overline{B_R})$, where $\overline{B_R} = \{u \in E \mid \|u\|_E \leq R\}$. Hence, taking into account (2.140) and the asymptotic semicompactness of m-semiflow G (see the proof of Theorem 2.9), we obtain that $\exists \{v_{n_k}(t_{n_k})\}_{k\geq 1} \subset \{v_n(t_n)\}_{n\geq 1}$, $\exists z \in \mathscr{A}$: $v_{n_k}(t_{n_k}) \to z$ in E, $k \to +\infty$. Further, $\forall k \geq 1$, we set $\varphi_k(t) = v_{n_k}(t_{n_k} + t)$, $t \in [0, M]$. Note that $\forall k \geq 1$ $\varphi_k(\cdot) \in \mathscr{D}_{0,M}(v_{n_k}(t_{n_k}))$. Then from Theorem 2.4, there exists a subsequence $\{\varphi_{k_j}\}_{j\geq 1} \subset \{\varphi_k\}_{k\geq 1}$ and an element $\varphi(\cdot) \in \mathscr{D}_{0,M}(z)$:

$$\varphi_{k_j} \to \varphi \text{ in } C([0, M]; E), \quad j \to +\infty. \tag{2.161}$$

At that, taking into account the invariance of $\mathscr{A}$ (see Theorem 2.9), $\forall t \in [0, M]$ $\varphi(t) \in \mathscr{A}$. In consequence of Lemma 2.25, there exist $y(\cdot), v(\cdot) \in \mathscr{K}$: $y(0) = z$, $v(0) = \varphi(M)$. For any $t \in \mathbf{R}$, we set

$$\psi(t) = \begin{cases} y(t), & t \leq 0, \\ \varphi(t), & t \in [0, M], \\ v(t - M), & t \geq M. \end{cases}$$

In view of Lemma 2.14, $\psi(\cdot) \in \mathscr{K}$. Therefore, from (2.160), we obtain:

$$\forall k \geq 1 \quad \|\Pi_M v_{n_k}(t_{n_k} + \cdot) - \Pi_M \psi(\cdot)\|_{C([0,M];E)} = \|\varphi_k - \varphi\|_{C([0,M];E)} \geq \varepsilon,$$

that contradicts with (2.161).

Thus, the set $\mathscr{P}$ constructed in (2.157) is the trajectory attractor in the trajectory space $\mathscr{K}_+$ with respect to the topology of $C^{loc}(\mathbf{R}_+; E)$.

2.7.3 Auxiliary Properties of the Global and Trajectory Attractors

Let $\mathscr{A}$ be a global attractor from Theorem 2.9 and $\mathscr{P}$ be a trajectory attractor from Theorem 2.10.

$$\mathscr{A} \text{ is a compact in the space } E \tag{2.162}$$

$$\mathscr{P} \text{ is a compact in the space } C^{loc}(\mathbf{R}_+; E) \tag{2.163}$$

Moreover,

$$\mathscr{P} = \Pi_+ \mathscr{K} = \Pi_+\{y \in \mathscr{K} \mid y(t) \in \mathscr{A} \ \forall t \in \mathbf{R}\}, \tag{2.164}$$

where $\mathscr{K}$ is the family of all complete trajectories of the inclusion (8) and Π_+ is the restriction operator on $\mathbf{R}_+$. Note that from Lemma 2.24, it follows that $\mathscr{K} \neq \emptyset$;

$$\mathscr{K} \text{ is a compact in the space } C^{loc}(\mathbf{R}; E); \tag{2.165}$$

$$\forall \xi(\cdot) \in \mathscr{K} \ \forall t \in \mathbf{R} \ \xi(t) \in \mathscr{A}; \tag{2.166}$$

$$\forall y_0 \in \mathscr{K} \ \forall t_0 \in \mathbf{R} \ \exists y(\cdot) \in \mathscr{K} : \ y(t_0) = y_0. \tag{2.167}$$

2.7.3.1 "Translation Compactness" of the Trajectory Attractor

For any $y \in \mathscr{K}$, let us set

$$\mathscr{H}(y) = \mathrm{cl}_{C^{loc}(\mathbf{R};E)}\{y(\cdot + s) \mid s \in \mathbf{R}\} \subset C^{loc}(\mathbf{R}; E) \cap L_\infty(\mathbf{R}; E).$$

Such family is said to be a hull of function $y(\cdot)$ in $\Xi = C^{loc}(\mathbf{R}; E)$.

Definition 2.3. The function $y(\cdot) \in \Xi$ is said to be translation compact (tr.-c.) in Ξ if the hull $\mathscr{H}(y)$ is compact in Ξ.

Similar constructions for the set of functional parameters that are responsible for nonautonomy of evolution equation are considered, for example, in [9, p. 917].

Definition 2.4. The family $\mathscr{U} \in \Xi$ is said to be translation compact, if $\mathscr{H}(\mathscr{U}) = \mathrm{cl}_\Xi\{y(\cdot + s) \mid \ y(\cdot) \in \mathscr{U}, s \in \mathbf{R}\}$ is a compact in Ξ.

From autonomy of system (8), applying the Arzelá-Ascoli compactness criterion (see the proof of Proposition 6.1 from [9]), we obtain the translation compactness criterion for the family $\mathscr{U}$:

(a) The set $\{y(t) \mid \ t \in \mathbf{R}, \ \ y \in \mathscr{U}\}$ is a compact in E.
(b) There exists a positive function $\alpha(s) \to 0+$ $(s \to 0+)$ such that

$$\|y(t_1) - y(t_2)\|_E \le \alpha(|t_1 - t_2|) \ \forall y \in \mathscr{U} \ \ \forall t_1, t_2 \in \mathbf{R}.$$

From the autonomy of problem (8) and (2.165), it follows that

Corollary 2.3. *$\mathscr{K}$ is translation-compact set in Ξ.*

Similarly, if we set $\Xi_+ = C^{loc}(\mathbf{R}_+; E)$ we obtain

Corollary 2.4. *$\mathscr{P}$ is translation-compact set in Ξ_+.*

2.7.3.2 Stability

Definition 2.5. [4, p. 487] The subset $\mathscr{A} \subset E$ is *Lyapunov stable* if for given $\varepsilon > 0$, there exists such $\delta > 0$ that if $D \subset E$ with $\mathrm{dist}(D, \mathscr{A}) < \delta$, then $\mathrm{dist}(T(t)D, \mathscr{A}) < \varepsilon$ for all $t \geq 0$.

We recall that (see [4, p.481])

$$T(t)D = \{\varphi(t) |\ \varphi(\cdot) \in \mathscr{D}(\varphi_0),\ \varphi_0 \in D\}.$$

Note also that

$$G(t, z) = T(t)\{z\} \quad \forall t \geq 0,\ \forall z \in E.$$

From [4, p. 487], it follows that a subset $\mathscr{A}$ is Lyapunov stable if and only if the given $\{\varphi_j(\cdot)\}_{j\geq 1}$ is a sequence of weak solutions (defined on $[0, +\infty)$) of problem (8) with $\mathrm{dist}(\varphi_j(0), \mathscr{A}) \to 0$, $j \to +\infty$ and $t_j \geq 0$ we have $\mathrm{dist}(\varphi_j(t_j), \mathscr{A}) \to 0$, $j \to +\infty$.

Definition 2.6. [4, p. 487] The subset $\mathscr{A}$ is uniformly asymptotically stable if $\mathscr{A}$ is Lyapunov stable and it is locally attracting (see [4, p. 482]).

Note that an attracting set is locally attracting one.

Corollary 2.5. *$\mathscr{A}$ is uniformly asymptotically stable.*

Proof. The proof follows from the definition of G, [4, Theorem 6.1], properties of solutions from Lemma 2.13, Theorem 2.4, and from the autonomy of problem (8).

Similar results are true for sets $\mathscr{P}$ and $\mathscr{K}$ in corresponding extended phase spaces.

2.7.3.3 Connectedness

Definition 2.7. [4, p. 485] M-semiflow G has Kneser's property, if $G(t, z)$ is connected for each $z \in E$, $t \geq 0$.

Corollary 2.6. *If G has Kneser's property, then $\mathscr{A}$ is connected.*

Proof. The proof follows from [4, Corollary 4.3], Lemma 2.13, and from the connectedness of the phase space E.

Note that the connectedness of G can be checked by different way (see, e.g., [34–36]). In order to do that, as a rule, it is required an auxiliary regularity of interaction functions. In the general case, Kneser's property for problem (8) can be checked using the method of proof from [36, Theorem 5], where we can consider Yosida approximation instead the proposed approximation.

Corollary 2.7. *If G has Kneser's property, then $\mathscr{K} \subset C^{loc}(\mathbf{R}; E)$ is connected and $\mathscr{P} \subset C^{loc}(\mathbf{R}_+; E)$ is connected.*

Proof. The proof is follows from (2.164) to (2.167) and from Corollary 2.6.

2.7.3.4 Behavior of Solutions on the Global Attractor

We say that the complete trajectory $\varphi \in \mathscr{K}$ is stationary if $\varphi(t) = z$ for all $t \in \mathbf{R}$ for some $z \in E$.

Following [4, p. 486], we denote the set of rest points of G by $Z(G)$.

Note that

$$Z(G) = \{(z, \bar{0}) |\ z \in B_0^{-1}(f - A_0(\bar{0}))\}.$$

Thus, $Z(G)$ is a convex, nonempty, weakly compact in $V \times V$ set.

For investigating of trajectory behavior of solutions on the attractor $\mathscr{A}$, it is necessary to consider similar definitions to [4, p. 486]:

Definition 2.8. A functional $\mathscr{V} : \mathscr{A} \to \mathbf{R}$ is a Lyapunov function for G on $\mathscr{A}$ provided

(i) $\mathscr{V}$ is continuous.
(ii) $\mathscr{V}(\varphi(t)) \leq \mathscr{V}(\varphi(s))$ whenever $\varphi \in \mathscr{K}$ and $t \geq s \geq 0$.
(iii) If $\mathscr{V}(\psi(t)) = constant$ for some $\psi \in \mathscr{K}$ and all $t \in \mathbf{R}$.

Then, ψ is stationary.

As a consequence of Theorem in the presence of a Lyapunov function, the behavior of such complete orbits can be characterized.

Theorem 2.11. *Suppose that there exists a Lyapunov function $\mathscr{V}$ for G on $\mathscr{A}$. Then for each $\psi \in \mathscr{K}$, the limit sets*

$$\alpha(\psi) = \{z \in E |\ \psi(t_j) \to z \text{ for some sequence } t_j \to -\infty\},$$

$$\omega(\psi) = \{z \in E |\ \psi(t_j) \to z \text{ for some sequence } t_j \to +\infty\}$$

are connected subsets of $Z(G)$ on which $\mathscr{V}$ is constant.

Proof. The proof follows from the proof of [4, Theorem 5.1], asymptotic compactness of G and from properties of solutions of problem (8).

2.7.3.5 The Estimate of the Fractal Dimension

As a rule, the finite dimension of the global attractor demands an auxiliary differentiability in some sense by initial data from the m-semiflow G that for one's turn involves an auxiliary regularity of interaction functions (in the case of problem (3)–(7), it involves an auxiliary regularity of functional j).

Let us show that, generally speaking, for problem (3)–(7), the fractal dimension of the attractor $\mathscr{A}$ can be equal to $+\infty$. In order to show that, we consider a particular case of problem (3)–(7). Let $N = 2$, $\Omega = (0,1) \times (0,\pi)$, $\Gamma_C = \{(x_1, x_2) | \; x_1 = 1, x_2 \in (0,\pi)\}$, $\Gamma_D = \partial\Omega \backslash \Gamma_C$.

First, we consider the auxiliary problem

$$\begin{cases} \triangle y = 0 \text{ in } \Omega, \\ y = 0 \text{ on } \Gamma_D, \\ \frac{\partial y}{\partial x_1} \in [-1,1] \text{ on } \Gamma_C. \end{cases} \tag{2.168}$$

For each $n \in \mathbf{N}$, let us set

$$y_n(x_1, x_2) = \frac{1}{n \cdot \cosh(n)} \sinh(n \cdot x_1) \cdot \sin(n \cdot x_2), \;\; (x_1, x_2) \in \bar{\Omega}.$$

Then $\forall c \in [-1,1]$ $\forall n \in \mathbf{N}$ $c \cdot y_n(\cdot)$ is a solution of (2.168).

Note that $\forall n \neq m$

$$(y_n, y_m)_{L_2(\Omega)} = 0,$$

$$\forall n \geq 1 \;\; \|y_n\|^2_{L_2(\Omega)} = \frac{\pi}{4n^3} \cdot \frac{1 - e^{-4n} - 4ne^{-2n}}{1 + e^{-4n} + 2e^{-2n}} \geq \frac{\bar{\alpha^*}^2}{n^3},$$

where $\bar{\alpha^*}$ does not depend on $n \in \mathbf{N}$.

In this case, if we set

$$z_n(\cdot) = \frac{\bar{\alpha^*}}{\|y_n\|_{L_2(\Omega)} \cdot n^{\frac{3}{2}}} \cdot y_n(\cdot), \;\; n \geq 1,$$

we obtain that the set

$$K = \{y \in L_2(\Omega) | \; y(x_1, x_2) = \sum_{k=1}^{\infty} \alpha_k z_k(x_1, x_2), \;\; (x_1, x_2) \in \Omega, \;\; \sum_{k=1}^{\infty} |\alpha_k| = 1\}$$

consists of solutions of problem (2.168).

For $N \geq 1$, we set $\varepsilon_N = \frac{\bar{\alpha^*}}{2N^{\frac{3}{2}+1}}$, $M(\varepsilon_N)$ is a minimal quantity of balls with radius ε_N, by the help of which we can cover K. Then

$$M(\varepsilon_N) \geq \bar{C} \cdot \frac{N^{\frac{3N}{2}}}{(N!)^3}$$

and

$$\forall d > 0 \;\; \overline{\lim_{N \to +\infty}} M(\varepsilon_N) \varepsilon_N^d = +\infty.$$

Therefore, the fractal dimension of the set K in the space $L_2(\Omega)$ as well as in the space $H^1(\Omega)$ is equal to $+\infty$.

Thus, the fractal dimension of the global attractor $\mathscr{A} \supset Z(G) \supset K \times K \times \{0\} \times \{0\}$ in the space E for the m-semiflow constructed on solutions of problem (3)–(7) in the case when $N = 2$,

$$B_0((y_1, y_2)^T) = (-\triangle y_1, -\triangle y_2),$$

Ω, Γ_D, Γ_C as in (2.168), $\partial j(x, 0) = [-1, 1]^2$ is equal to $+\infty$. Thus, we can see that the dimension of the attractor in the given case sufficiently depends on the differentiability of the functional $j(x, u)$ for $u = 0$.

2.8 Applications

As applications, we can consider new classes of problems with degenerations, problems on a manifold, problems with delay, stochastic partial differential equations, etc., [2–5, 7, 9–39, 41, 42] with differential operators of pseudomonotone type as corresponding choice of the phase space. Let us consider some particular classes of examples, when we can obtain stronger results for resolving operator.

2.8.1 Climate Energy Balance Model

We now consider a climate energy balance model (see Example 4). The problem is the following:

$$\begin{array}{ll} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} + Bu \in QS(x)\beta(u) + h(x), & (t, x) \in \mathbf{R}_+ \times (-1, 1), \\ u_x(-1, t) = u_x(1, t) = 0, & t \in \mathbf{R}_+, \\ u(x, 0) = u_0(x), & x \in (-1, 1), \end{array} \tag{2.169}$$

where B and Q are positive constants, $S, h \in L_\infty(-1, 1)$, $u_0 \in L_2(-1, 1)$, and β is a maximal monotone graph in $\mathbf{R}^2$, which is bounded, that is, there exist $m, M \in \mathbf{R}$ such that

$$m \le z \le M, \quad \text{for all } z \in \beta(s), s \in \mathbf{R}. \tag{2.170}$$

We also assume that

$$0 < S_0 \le S(x) \le S_1, \quad \text{a.e. } x \in (-1, 1). \tag{2.171}$$

The unknown $u(t, x)$ represents the averaged temperature of the Earth surface, Q is the so-called solar constant, which is the average (over a year and over the surface of the Earth) value of the incoming solar radiative flux, and the function $S(x)$ is the insolation function given by the distribution of incident solar radiation at the top of the atmosphere. When the averaging time is of the order of 1 year or longer, the function $S(x)$ satisfies (2.171); for shorter periods, we must assume that $S_0 = 0$. The term β represents the so-called co-albedo function, which can be possibly discontinuous. It represents the ratio between the absorbed solar energy and the incident solar energy at the point x on the Earth surface. Obviously, $\beta(u(x,t))$ depends on the nature of the Earth surface. For instance, it is well known that on ice sheets, $\beta(u(x,t))$ is much smaller than on the ocean surface because the white color of the ice sheets reflects a large portion of the incident solar energy, whereas the ocean, due to its dark color and high heat capacity, is able to absorb a larger amount of the incident solar energy. We point out that this model is the particular case of the first-order evolution inclusion, considered in Sects. 2.3 and 2.4. All results from this subsection are fulfilled for state function of this problem.

2.8.2 Application for General Classes High-Order Nonlinear PDEs

Consider an example of the class of nonlinear boundary-value problems for which we can investigate the dynamics of solutions as $t \to +\infty$. Note that in discussion, we do not claim generality.

Let $n \geq 2$, $m \geq 1$, $p \geq 2$, $1 < q \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\Omega \subset \mathbf{R}^n$ be a bounded domain with rather smooth boundary $\Gamma = \partial\Omega$. We denote a number of differentiations by x of order $\leq m-1$ (correspondingly of order $= m$) by N_1 (correspondingly by N_2). Let $A_\alpha(x, \eta; \xi)$ be a family of real functions ($|\alpha| \leq m$), defined in $\Omega \times \mathbf{R}^{N_1} \times \mathbf{R}^{N_2}$ and satisfying the next properties:

(C_1) For a.e. $x \in \Omega$ the function $(\eta, \xi) \to A_\alpha(x, \eta, \xi)$ is continuous one in $\mathbf{R}^{N_1} \times \mathbf{R}^{N_2}$.

(C_2) $\forall (\eta, \xi) \in \mathbf{R}^{N_1} \times \mathbf{R}^{N_2}$ the function $x \to A_\alpha(x, \eta, \xi)$ is measurable one in Ω.

(C_3) Exist such $c_1 \geq 0$ and $k_1 \in L_q(\Omega)$ that for a.e. $x \in \Omega$, $\forall (\eta, \xi) \in \mathbf{R}^{N_1} \times \mathbf{R}^{N_2}$

$$|A_\alpha(x, \eta, \xi)| \leq c_1[|\eta|^{p-1} + |\xi|^{p-1} + k_1(x)].$$

(C_4) Exist such $c_2 > 0$ and $k_2 \in L_1(\Omega)$ that for a.e. $x \in \Omega$, $\forall (\eta, \xi) \in \mathbf{R}^{N_1} \times \mathbf{R}^{N_2}$

$$\sum_{|\alpha|=m} A_\alpha(x, \eta, \xi)\xi_\alpha \geq c_2|\xi|^p - k_2(x).$$

(C_5) For a.e. $x \in \Omega$, $\forall \eta \in \mathbf{R}^{N_1}$, $\forall \xi, \xi^* \in \mathbf{R}^{N_2}$, $\xi \neq \xi^*$, the inequality

$$\sum_{|\alpha|=m} (A_\alpha(x, \eta, \xi) - A_\alpha(x, \eta, \xi^*))(\xi_\alpha - \xi^*_\alpha) > 0$$

takes place.

Consider such denotations: $D^k u = \{D^\beta u, |\beta| = k\}$, $\delta u = \{u, Du, \ldots, D^{m-1}u\}$ (see [22, c. 194]).

For an arbitrary fixed interior force $f \in L_2(\Omega)$, we investigate the dynamics of all weak (generalized) solutions defined on $[0, +\infty)$ of such problem:

$$\frac{\partial y(x,t)}{\partial t} + \sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha \left(A_\alpha(x, \delta y(x,t), D^m y(x,t))\right) = f(x) \text{ on } \Omega \times (0, +\infty), \tag{2.172}$$

$$D^\alpha y(x,t) = 0 \text{ on } \Gamma \times (0, +\infty), \quad |\alpha| \le m-1. \tag{2.173}$$

as $t \to +\infty$.

Consider such denotations (see for detail [22, c. 195]): $H = L_2(\Omega)$, $V = W_0^{m,p}(\Omega)$ is a real Sobolev space,

$$a(u, \omega) = \sum_{|\alpha| \le m} \int_\Omega A_\alpha(x, \delta u(x), D^m u(x)) D^\alpha \omega(x) dx, \quad u, \omega \in V.$$

Note that Condition (H_2) takes place in view of Sobolev theorem on compactness of embedding. Taking into account Conditions (C_1)–(C_5) and [22, p. 192–199], the operator $A : V \to V^*$, defined by the formula $\langle A(u), \omega \rangle_V = a(u, \omega)$ $\forall u, \omega \in V$, satisfies Conditions (A_1)–(A_3). Hence, we can pass from problem (2.172)–(2.173) to corresponding problem in "generalized" setting (6.5). Note that

$$A(u) = \sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha \left(A_\alpha(x, \delta u, D^m u)\right) \quad \forall u \in C_0^\infty(\Omega).$$

Therefore, all statements from previous subsections, in particular, Theorems 2.1–2.3 and Lemmas 2.5–2.11, are fulfilled for weak (generalized) solutions of problem (2.172)–(2.173).

Remark 2.4. As applications, we can also consider new classes of problems with degenerations, problems on a manifold, problems with delay, stochastic partial differential equations, etc. [10, 14, 22, 32], with differential operators of pseudomonotone type as corresponding choice of the phase space.

2.8.3 Application for Chemotaxis Processes

Let us consider the problem from Example 5. This problem connected with the movement of biological cells or organisms in response to chemical gradients. If properly interpreting the derivative and correspondingly choosing phase spaces, all models can be reduced to the first-order autonomous evolution equation. For example, let us consider a particular case and examine asymptotical behavior of solutions. We consider the problem that described by the following initial-boundary problem for a quasi-linear parabolic system of equations

$$\begin{aligned} &\frac{\partial \rho}{\partial t} = b\Delta\sigma - c\rho + du \quad \text{in} \quad \Omega \times (0,\infty), \\ &\frac{\partial u}{\partial n} = \frac{\partial \rho}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty), \\ &u(x,0) = u_0(x), \quad \rho(x,0) = \rho_0(x) \quad \text{in} \quad \Omega. \end{aligned} \tag{2.174}$$

Here, $u(x,t)$ and $\rho(x,t)$ denote the population density of biological individuals and the concentration of chemical substance at a position $x \in \Omega \subset \mathbf{R}^2$ and a time $t \in [0,\infty)$, respectively. The mobility of individuals consists of two effects: one is random walking and the other is the directed movement in a sense that they have a tendency to move toward higher concentration of the chemical substance. This is called chemotaxis in biology [1,7,13,27]. $a > 0$ and $b > 0$ are the diffusion rates of u and ρ, respectively. $c > 0$ and $d > 0$ are the degradation and production rates of ρ, respectively. $\chi(\rho)$ is the sensitivity function due to chemotaxis. It is a real smooth function of $\rho \in [0,\infty)$ with uniformly bounded derivatives up to the third-order

$$\sup_{\rho \geq 0} \left| \frac{d^i \chi}{d\rho^i}(\rho) \right| < \infty \quad \text{for} \quad i = 1,2,3. \tag{2.175}$$

$f(u)$ is a growth term of u. It is a real smooth function of $u \in [0,\infty)$ such that $f(0) = 0$ and

$$f(u) = (-\mu u + \nu)u \quad \text{for sufficiently large } u \tag{2.176}$$

with $\mu > 0$ and $-\infty < \nu < \infty$. Let $f(u) = f_1(u)u$, then $f_1(u)$ is a smooth function of $u \in [0,\infty)$ such that $f_1(u) = -\mu u + \nu$ for sufficiently large u.

For the abstract setting of the problem, we set the product space $H = L_2(\Omega) \times H^1(\Omega)$, and consider (2.174) as an initial value problem of an evolution equation

$$\begin{aligned} &\frac{dU}{dt} + AU = F(U), \quad 0 < t < \infty, \\ &U(0) = U_0 \end{aligned} \tag{2.177}$$

in H. Here, A and $F(U)$ are defined by

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \quad \text{with} \quad \mathscr{D}(A) = H_N^2(\Omega) \times H_N^3(\Omega),$$

where $A_1 = -a\triangle + 1$ and $A_2 = -b\triangle + c$, and

$$F(U) = \begin{pmatrix} -\nabla\{u\nabla\chi(\rho)\} + \{1 + f_1(u)\}u \\ du \end{pmatrix}, \quad U = \begin{pmatrix} u \\ \rho \end{pmatrix} \in \mathscr{D}(A).$$

The set of initial values is set by

$$K = \left\{ \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in L_2(\Omega) \times H^{1+\varepsilon_0}(\Omega) : u_0 \geq 0, \ \rho_0 \geq 0 \right\}$$

where $0 < \varepsilon_0 < \frac{1}{2}$ is some fixed exponent and $U_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix}$ is in K.

In [29], it is proved that there exists a unique global solution $U = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to (2.177) and that the solution is continuous with respect to the initial value. Therefore, a continuous semigroup $\{S(t)\}_{t\geq 0}$ can be defined on K by $S(t)U_0 = U(t)$. For $t > 0$, $S(t)$ maps K into $K \cap \mathscr{A}$.

Proposition 2.6. *[29] There exists a universal constant $C > 0$ such that the following statement holds for each bounded ball $B_r = \left\{ \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in K : \|u_0\|_{L_2} + \|\rho_0\|_{H^{1+\varepsilon_0}} \leq r \right\}$, there exists a time $t_r > 0$ depending on B_r such that*

$$\sup_{t\geq t_r} \sup_{U_0 \in B_r} \|S(t)U_0\|_{H^2 \times H^3} \leq C.$$

The set $\mathscr{B} = \left\{ \begin{pmatrix} u \\ \rho \end{pmatrix} \in H^2(\Omega) \times H^3(\Omega); \ \|u\|_{H^2} + \|\rho\|_{H^3} \leq C \right\} \cap K$, where C is the constant appearing in Proposition 2.6, is a compact absorbing set for $(\{S(t)\}_{t\geq 0}, K)$. Hence, by virtue of [33, Chap. 1, Theorem 1.1], there exists a global attractor $\mathscr{A} \subset K$, $\mathscr{A}$ being a compact and connected subset of K.

We set

$$\mathscr{X} = \overline{\bigcup_{t\geq t_{\mathscr{B}}} S(t)\mathscr{B}} \quad \text{(closure in the topology of} \quad K)$$

using a time $t_{\mathscr{B}}$ such that $S(t)\mathscr{B} \subset \mathscr{B}$ for every $t \geq t_{\mathscr{B}}$. We note that $\mathscr{X}$ is a compact set of K with the relation $\mathscr{A} \subset \mathscr{X} \subset \mathscr{B}$ and is absorbing and positively invariant for $\{S(t)\}_{t\geq 0}$.

Definition 2.9. A subset $\mathscr{M} \subset \mathscr{X}$ is called the *exponential attractor* for $(\{S(t)\}_{t\geq 0}, \mathscr{X})$ if (i) $\mathscr{A} \subset \mathscr{M} \subset \mathscr{X}$, (ii) $\mathscr{M}$ is a compact subset of H and is a positively invariant set for $S(t)$, (iii) $\mathscr{M}$ has finite fractal dimension $d_F(\mathscr{M})$, and (iv) $h(S(t)\mathscr{X}, \mathscr{M}) \leq c_0 exp(-c_1 t)$ for $t \geq 0$ with some constants $c_0, c_1 > 0$, where

$$h(B_0, B_1) = \sup_{U\in B_0} \inf_{V \in B_1} \|U - V\|_H$$

denotes the Hausdorf pseudodistance of two sets B_0 and B_1.

Now, it is sufficiently to apply the next theorem to the dynamical system $(\{S(t)\}_{t\geq 0}, \mathscr{X})$.

Theorem 2.12. *[29] Let $F(U)$ satisfy the Lipschitz condition*

$$\|F(U) - F(V)\|_H \leq C\|A^{1/2}(U - V)\|_H, \quad U, V \in \mathscr{X} \tag{2.178}$$

and let the mapping $G(t, U_0) = S(t)U_0$ from $[0, T]\times\mathscr{X}$ into $\mathscr{X}$ satisfy the Lipschitz condition

$$\|G(t, U_0) - G(s, V_0)\|_H \leq C_T\{|t - s| + \|U_0 - V_0\|_H\}, \quad t, s \in [0, T], U_0, V_0 \in \mathscr{X} \tag{2.179}$$

for each $T > 0$.

Then there exists an exponential attractor $\mathscr{M}$ for $(\{S(t)\}_{t\geq 0}, \mathscr{X})$.

Thus, we arrive at the main result. This result is borrowed from [29].

Theorem 2.13. *There exists an exponential attractor $\mathscr{M}$ of the dynamical system $(\{S(t)\}_{t\geq 0}, \mathscr{X})$ in H.*

Remark 2.5. For simplicity, we have assumed the condition (2.176) on $f(u)$. But this is not essential; indeed, the theorem can be proved under the conditions that

$$-\eta'(u^m + 1)u \leq f(u) \leq (-\mu' u + \nu')u, \quad u \geq 0,$$
$$-\eta'(u^m + 1)u \leq f'(u)u \leq (-\mu' u + \nu')u, \quad u \geq 0$$

with some constants $\mu', \nu', \eta' > 0$ and some positive integer m.

Remark 2.6. If the stronger decay condition

$$f(u) = -\mu u^3 + \nu u^2 + \lambda u \quad \text{for sufficiently large } u \tag{2.180}$$

with $\mu > 0$ and $-\infty < \nu, \lambda < \infty$, is assumed instead of (2.176), then $\chi(\rho)$ is allowed only to satisfy

$$\left|\frac{d^i\chi}{d\rho^i}(\rho)\right| \leq \chi_0(\rho^m + 1), \quad \rho \geq 0 \quad \text{for} \quad i = 1, 2, 3 \tag{2.181}$$

with some constant χ_0 and some positive integer m. For example, a sensitivity function $\chi(\rho) = \chi_0\rho^2$ can be taken.

2.8.4 Applications for Damped Viscoelastic Fields with Short Memory

We consider a linear viscoelastic body occupying the bounded domain Ω in $\mathbf{R}^N$ $(N = 2, 3)$ in a strainless state which is acted upon by volume forces and surface tractions and which may come in contact with a foundation on the part Γ_C of the boundary $\partial\Omega$ (see Example 1). The boundary $\partial\Omega$ of the set Ω is supposed to be a regular one, and point data of $x \in \bar{\Omega}$ is considered in some fixed Cartesian system of coordinates. We assume that the body is endowed with short memory, that is, the state of the stress at the instant t depends only on the strain at the instant t and at the immediately preceding instants. In this case, the equation of state has the next form:

$$\sigma_{ij}(u) = b_{ijhk}\varepsilon_{kh}(u) + a_{ijhk}\frac{\partial}{\partial t}\varepsilon_{kh}(u), \quad i, j = 1, \ldots, N, \tag{2.182}$$

where $u : \Omega \times (0, +\infty) \to \mathbf{R}^N$ denotes the displacement field, $\sigma = \sigma(u)$ is the stress tensor, and $\varepsilon = \varepsilon(u)$ is the strain tensor, $\varepsilon_{hk}(u) = \frac{1}{2}(u_{k,h} + u_{h,k})$. The viscosity coefficients a_{ijhk} and the elasticity coefficients b_{ijhk} satisfy the well-known symmetry and ellipticity conditions. The dynamic behavior of the body is described by the equilibrium equation:

$$\sigma_{ij,j}(u) + f_i = \frac{\partial^2}{\partial t^2}u_i \quad \text{in} \quad \Omega \times (0, +\infty), \tag{2.183}$$

where $f = \{f_i\}_{i=1}^N \in L_2(\Omega; \mathbf{R}^N)$ denotes the density of body force. We suppose that the boundary $\partial\Omega$ is divided into three parts: Γ_D, Γ_N, and Γ_C. Exactly, let Γ_D, Γ_N, and Γ_C be disjoint sets and $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N} \cup \overline{\Gamma_C}$. We assume that $\Gamma_C \subset \partial\Omega$ is an open subset with positive surface measure (cf. [30, p. 196]). The displacements

$$u_i = 0 \text{ on } \Gamma_D \times (0, +\infty), \tag{2.184}$$

are prescribed on Γ_D, and surface tractions

$$S_i = \sigma_{ij}n_j = F_i \ \ (F_i = F_i(x)) \text{ on } \Gamma_N \times (0, +\infty). \tag{2.185}$$

are prescribed on Γ_N, where $F = \{F_i\}_{i=1}^N \in L_2(\Gamma_N; \mathbf{R}^N)$ denotes the vector of surface traction, $S = \{S_i\}_{i=1}^N$ is the stress vector on Γ_N, and $n = \{n_i\}_{i=1}^N$ is the outward unit normal to $\partial\Omega$.

On Γ_C, we specify nonmonotone multivalued boundary "reaction-velocity" conditions:

$$-S \in \partial j\left(x, \frac{\partial u}{\partial t}\right) \text{ on } \Gamma_C \times (0, +\infty), \tag{2.186}$$

where $j : \Gamma_C \times \mathbf{R}^N \to \mathbf{R}$ satisfies the next conditions:

1. $j(\cdot, \xi)$ is a measurable function for each $\xi \in \mathbf{R^N}$ and $j(\cdot, 0) \in L_1(\Gamma_C)$.
2. $j(x, \cdot)$ is a locally Lipschitz function for each $x \in \Gamma_C$.
3. $\exists \bar{c} > 0$: $\|\eta\|_{\mathbf{R}^N} \le \bar{c}(1 + \|\xi\|_{\mathbf{R}^N})$ $\forall x \in \Gamma_C$, $\forall \xi \in \mathbf{R}^N$, $\forall \eta \in \partial j(x, \xi)$, where for $x \in \Gamma_C$,

$$\partial j(x,\xi) = \{\eta \in \mathbf{R}^N \mid (\eta, v)_{\mathbf{R}^N} \le j^0(x,\xi; v)\ \forall v \in \mathbf{R}^N\}$$

is the generalized gradient of the functional $j(x,\cdot)$ at point $\xi \in \mathbf{R}^N$ and

$$j^0(x,\xi;v) = \varlimsup_{\zeta\to\xi,\, t\searrow 0} \frac{j(x,\zeta+tv) - j(x,\zeta)}{t}$$

is the generalized upper derivative of $j(x,\cdot)$ at point $\xi \in \mathbf{R}^N$ and the direction $v \in \mathbf{R}^N$.

Note that all nonconvex superpotential graphs from, in particular, the functions j, defined as a minimum and as a maximum of quadratic convex functions, satisfy the upper considered conditions on Γ_C.

For the variational formulation of the problem (2.182)–(2.186), we set: $H = L_2(\Omega;\mathbf{R}^N)$, $Z = H^\delta(\Omega;\mathbf{R}^N)$, $V = \{v \in H^1(\Omega;\mathbf{R}^N) : v_i = 0 \text{ on } \Gamma_D\}$, where $\delta \in \left(\frac{1}{2};1\right)$. Let $\forall u, v \in V$

$$\langle f_0, v\rangle_V = \int_\Omega f_i v_i dx + \int_{\Gamma_N} F_i v_i d\sigma(x),$$

$$a(u,v) = \int_\Omega a_{ijhk}\varepsilon_{ij}(u)\varepsilon_{ij}(v)dx,$$

$$b(u,v) = \int_\Omega b_{ijhk}\varepsilon_{ij}(u)\varepsilon_{ij}(v)dx,$$

$\bar\gamma : Z \to L_2(\partial\Omega;\mathbf{R}^N)$ be a trace operator and $\bar\gamma^* : L_2(\partial\Omega;\mathbf{R}^N) \to Z^*$ be a conjugate operator,

$$\bar\gamma^* u(z) = \int_{\partial\Omega} u(x)\bar\gamma z(x) d\sigma(x), \quad z \in Z,\ u \in L_2(\partial\Omega;\mathbf{R}^N).$$

Let us consider a locally Lipschitz functional $J : L_2(\Gamma_C;\mathbf{R}^n) \to \mathbf{R}$,

$$J(z) = \int_{\Gamma_C} j(x, z(x)) d\sigma(x), \quad z \in L_2(\Gamma_C;\mathbf{R}^n).$$

Then the interaction functions A_1, A_2, and B_0 can be defined by the next relations:

$$\forall z \in Z \quad A_2(z) = \bar\gamma^*\left(\partial J(\bar\gamma z)\right),$$

$$\forall u, v \in V \quad \langle A_1 u, v\rangle_V = a(u,v), \quad \langle B_0 u, v\rangle_V = b(u,v),\ A_0(u) = A_1 u + A_2(u).$$

If we supplementary have $\bar{\alpha} > \bar{c}\bar{\beta}^2\|\bar{\gamma}\|^2$, where $\bar{\beta}$ is the embedding constant of V into Z, $\bar{\alpha}$ is the constant from the ellipticity condition for a_{ijhk}, or

$$\forall x \in \Gamma_C,\ \forall \xi \in \mathbf{R}^N,\ \forall \eta \in \partial j(x,\xi) \quad (\eta,\xi)_{\mathbf{R}^N} \geq 0,$$

then from [25], it follows that the next condition hold true:

(H_1) V, Z, H are Hilbert spaces; $H^* \equiv H$ and we have such chain of dense and compact embeddings:

$$V \subset Z \subset H \equiv H^* \subset Z^* \subset V^*.$$

(H_2) $f_0 \in V^*$.
(A_1) $\exists c > 0 : \forall u \in V, \forall d \in A_0(u)\ \|d\|_{V^*} \leq c(1 + \|u\|_V)$.
(A_2) $\exists \alpha, \beta > 0 : \forall u \in V, \forall d \in A_0(u)\ \langle d, u\rangle_V \geq \alpha\|u\|_V^2 - \beta$.
(A_3) $A_0 = A_1 + A_2$, where $A_1 : V \to V^*$ is linear, selfconjugated, positive operator and $A_2 : V \overset{\to}{\to} V^*$ satisfies such conditions:

(a) There exists such Hilbert space Z that the embedding $V \subset Z$ is dense and compact one and the embedding $Z \subset H$ is dense and continuous one.
(b) For any $u \in Z$, the set $A_2(u)$ is nonempty, convex, and weakly compact one in Z^*.
(c) $A_2 : Z \overset{\to}{\to} Z^*$ is a bounded map, that is, A_2 converts bounded sets from Z into bounded sets in the space Z^*.
(d) $A_2 : Z \overset{\to}{\to} Z^*$ is a demiclosed map, that is, if $u_n \to u$ in Z, $d_n \to d$ weakly in Z^*, $n \to +\infty$, and $d_n \in A_2(u_n)\ \forall n \geq 1$, then $d \in A_2(u)$.

(B_1) $B_0 : V \to V^*$ is a linear selfconjugated operator.

(B_2) $\exists \gamma > 0 : \langle B_0 u, u\rangle_V \geq \gamma\|u\|_V^2$.
Here, $\langle \cdot, \cdot \rangle_V : V^* \times V \to \mathbf{R}$ is the duality in $V^* \times V$, coinciding on $H \times V$ with the inner product $(\cdot, \cdot)$ in Hilbert space H.

Note that from (A_1)–(A_3) and results of this chapter, it follows that the map A_0 satisfies such condition:
$(A_3)'$ $A_0 : V \overset{\to}{\to} V^*$ is (generalized) λ_0-pseudomonotone, that is:

(a) For any $u \in V$, the set $A_0(u)$ is nonempty, convex, and weakly compact one in V^*.
(b) If $u_n \to u$ weakly in V, $n \to +\infty$, $d_n \in A_0(u_n)\ \forall n \geq 1$ and $\overline{\lim\limits_{n\to\infty}} \langle d_n, u_n - u\rangle_V \leq 0$, then $\forall \omega \in V\ \exists d(\omega) \in A_0(u)$:

$$\underline{\lim_{n\to+\infty}} \langle d_n, u_n - \omega\rangle_V \geq \langle d(\omega), u - \omega\rangle_V.$$

(c) The map A_0 is upper semicontinuous one that acts from an arbitrary finite-dimensional subspace of V into V^*, endowed with weak topology.

Thus, in the next chapter, we investigate the dynamic of all weak solutions of the second-order nonlinear autonomous differential-operator inclusion

$$y''(t) + A_0(y'(t)) + B_0(y(t)) \ni f_0, \tag{2.187}$$

as $t \to +\infty$, which are defined as $t \geq 0$, where parameters of the problem satisfy conditions (H_1), (H_2), (A_1)–(A_3) and (B_1)–(B_2).

As a *weak solution* of the evolution inclusion (2.187) on the interval $[\tau, T]$, we consider such pair of elements $(u(\cdot), u'(\cdot))^T \in L_2(\tau, T; V \times V)$ that for some $d(\cdot) \in L_2(\tau, T; V^*)$

$$\begin{aligned} & d(t) \in A_0(u'(t)) \quad \text{for almost every (a.e.)} \quad t \in (\tau, T), \\ & -\int_\tau^T (\zeta'(t), u'(t))dt + \int_\tau^T \langle d(t), \zeta(t)\rangle_V dt + \int_\tau^T \langle B_0 u(t), \zeta(t)\rangle_V dt \\ & \quad = \int_\tau^T \langle f_0, \zeta(t)\rangle_V \quad \forall \zeta \in C_0^\infty([\tau, T]; V), \end{aligned} \tag{2.188}$$

where u' is the derivative of the element $u(\cdot)$ in the sense of the space of distributions $\mathscr{D}^*([\tau, T]; V^*)$.

As a *generalized solution* of the problem (2.182)–(2.186), we consider the weak solution of the corresponding problem (2.187). All results from Sects. 2.5 and 2.7 for state functions of this problem are fulfilled.

Corollary 2.8. *The m-semiflow G constructed on all generalized solutions of (2.182)–(2.186) has the compact invariant global attractor $\mathscr{A}$. For all generalized solutions (2.182)–(2.186), defined on $[0, +\infty)$, there exists the trajectory attractor $\mathscr{P}$. At that,*

$$\mathscr{A} = \mathscr{P}(0) = \{y(0) \mid y \in \mathscr{K}\}, \quad \mathscr{P} = \Pi_+ \mathscr{K},$$

where $\mathscr{K}$ is the family of all complete trajectories of corresponding differential-operator inclusion in $C^{loc}(\mathbf{R}; E) \cap L_\infty(\mathbf{R}; E)$. Moreover, global attractors are equal in the sense of [24, Definition 6, p. 88] as well as in the sense of [37, Definition 2.2, p. 182].

Thus, all statements of previous sections hold true for all generalized solutions of problem (2.182)–(2.186). In particular, all displacements and velocities are "attracted" as $t \to +\infty$ to all complete (defined on the entire time axis), globally bounded trajectories of corresponding "generalized" problem, which belongs to compact sets in corresponding phase and extended phase spaces. Questions concerning the connection and dimension of constructed attractors in the general case are opened. Note that approaches proposed in works [24, 37] are based on properties of solutions of evolution objects. Our approaches are based on properties of interaction function A from inclusion and properties of phase spaces.

2.8.5 Applications for Nonsmooth Autonomous Piezoelectric Fields

We consider a mathematical model which describes the contact between a piezoelectric body and a foundation (see Example 2). The physical setting is formulated as in [23]. We consider a plane electro-elastic material which in its unreformed state occupies an open bounded subset Ω of $\mathbf{R}^d$, $d = 2$. We agree to keep this notation since the mathematical results hold for $d \geq 2$. The boundary Γ of the piezoelectric body Ω is assumed to be Lipschitz continuous. We consider two partitions of Γ. A first partition is given by two disjoint measurable parts Γ_D and Γ_N such that $m(\Gamma_D) > 0$, and a second one consists of two disjoint measurable parts Γ_a and Γ_b such that $m(\Gamma_a) > 0$. We suppose that the body is clamped on Γ_D, so the displacement field vanishes there. Moreover, a surface tractions of density g act on Γ_N, and the electric potential vanishes on Γ_a.

The body Ω is lying on another medium (the so-called support) which introduce frictional effects. The interaction between the body and the support is described, due to the adhesion or skin friction, by a nonmonotone possibly multivalued law between the bonding forces and the corresponding displacements. In order to formulate the skin effects, we suppose that the body forces of density f consist of two parts: f_1 which is prescribed external loading and f_2 which is the reaction of constrains introducing the skin effects, that is, $f = f_1 + f_2$. Here, f_2 is a possibly multivalued function of the displacement u. We consider the reaction-displacement law of the form

$$-f_2(x) \in \partial j(x, u(x,t)) \text{ in } \Omega,$$

where $j : \Omega \times \mathbf{R}^2 \to \mathbf{R}$ is locally Lipschitz function in its last variable and ∂j represents the Clarcke subdifferential.

The governing equations of piezoelectricity consist (see Example 2) of the equation of motion, equilibrium equation, constitutive relations, strain-displacement, and electric field-potential relations. We suppose that the accelerations in the system are not negligible, and therefore, the process is dynamic.

The equation of motion for the stress field and *the equilibrium equation* for the electric displacement field are, respectively, given by

$$\rho u'' - \mathrm{Div}\sigma = \rho f - \gamma u' \text{ in } \Omega \times (0, +\infty),$$

$$\mathrm{div} D = 0 \text{ in } \Omega \times (0, +\infty),$$

where ρ is the constant mass density (normalized as $\rho = 1$), $\gamma \in L_\infty(\Omega)$, $\gamma(x) \geq \gamma_0 > 0$ for a.e. $x \in \Omega$ is a nonnegative function characterizing the viscosity (damping) of the medium, and $\sigma : \Omega \times (0, +\infty) \to S_d$, $\sigma = (\sigma_{ij})$, and $D : \Omega \to \mathrm{R}^d$, $D = (D_i)$, $i, j = 1, \ldots, d$ represent the stress tensor and the electric displacement field, respectively. Here, S_d is the linear space of second-order symmetric tensors on R^d with the inner product and the corresponding norm $\sigma : \tau = \sum_{ij} \sigma_{ij}\tau)ij$, $\|\tau\|^2_{S_d} = \tau : \tau$, respectively. Recall also that Div is the divergence operator for tensor valued functions given by $\mathrm{Div}\sigma = (\sigma_{ij,j})$ and div stands for the divergence operator for vector-valued functions, that is, $\mathrm{div}D = (D_{i,i})$.

The stress-charge form of piezoelectric constitutive relations describes the behavior of the material and are the following:

$$\sigma = \mathscr{A}\varepsilon(u) - \mathscr{P}E(\varphi) \text{ in } \Omega \times (0,+\infty) \text{ (converse effect)},$$
$$D = \mathscr{P}\varepsilon(u) + \mathscr{B}E(\varphi) \text{ in } \Omega \times (0,+\infty) \text{ (direct effect)},$$

where $\mathscr{A} : \Omega \times S_d \to S_d$ is a linear elasticity operator with the elasticity tensor $a = (a_{ijkl})$, $\mathscr{P} : \Omega \times S_d \to \mathbf{R}^d$ is a linear piezoelectric operator represented by the piezoelectric coefficients $p = (p_{ijk})$, $i, j, k = 1, \ldots d$ (third order tensor field), $\mathscr{P}^T : \Omega \times \mathbf{R}_d \to S_d$ is its transpose represented by $p^T = (p^T_{ijk}) = (p_{kij})$, and $\mathscr{B} : \Omega \times \mathbf{R}^d \to \mathbf{R}^d$ is a linear electric permittivity operator with the dielectric constants $\beta = (\beta_{ij})$ (second order tensor field). The decoupled state (purely elastic and purely electric deformations) can be obtained by setting the piezoelectric coefficients $p_{ijk} = 0$. The elasticity coefficients $a(x) = (a_{ijkl}(x))$, $i, j, k, l = 1, \ldots d$ (fourth-order tensor field) are functions of position in a nonhomogeneous material. We use here notation p^T to denote the transpose of the tensor p given $p\tau \cdot v = \tau : p^T v$ for $\tau \in S_d$ and $v \in \mathbf{R}^d$.

The elastic strain-displacement and electric field-potential relations are given by

$$\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T) \text{ in } \Omega \times (0,+\infty),$$
$$E(u) = -\nabla\varphi \text{ in } \Omega,$$

where $\varepsilon(u) = (\varepsilon_{ij}(u))$ and $E(\varphi) = (E_i(\varphi))$ denote the linear strain tensor and the electric vector field, respectively. Here, $u : \Omega \times (0,+\infty) \to \mathbf{R}^d$, $u = (u_i)$, $i = 1, \ldots, d$ and $\varphi : \Omega \to \mathbf{R}$ are the displacement vector field and the electric potential (scalar field), respectively.

Denoting by u_0 and u_1, the initial displacement and initial velocity, respectively, the classical formulation of the mechanical model can be stated as follows: find a displacement field $u : \Omega \times (0,+\infty) \to \mathbf{R}^d$ and an electric potential $\varphi : \Omega \to \mathbf{R}^d$ such that

$$u'' - \mathrm{Div}\sigma = f_1 + f_2 - \gamma u' \text{ in } \Omega \times (0,+\infty) \tag{2.189}$$

$$\mathrm{div}D = 0 \text{ in } \Omega \times (0,+\infty) \tag{2.190}$$

$$\sigma = \mathscr{A}\varepsilon(u) + \mathscr{P}^T\nabla\varphi \text{ in } \Omega \times (0,+\infty), \tag{2.191}$$

$$D = \mathscr{P}\varepsilon(u) - \mathscr{B}\nabla\varphi \text{ in } \Omega \times (0,+\infty), \tag{2.192}$$

$$u = 0 \text{ on } \Gamma_D \times (0,+\infty) \tag{2.193}$$

$$\sigma n = g \text{ on } \Gamma_D \times (0,+\infty) \tag{2.194}$$

$$\varphi = 0 \text{ on } \Gamma_a \times (0,+\infty) \tag{2.195}$$

$$D \cdot n = 0 \text{ on } \Gamma_b \times (0, +\infty) \tag{2.196}$$

$$-f_2(x) \in \partial j(x, u(x,t)) \text{ in } \Omega \times (0, +\infty) \tag{2.197}$$

$$u(0) = u_0, \ u'(0) = u_1 \text{ in } \Omega, \tag{2.198}$$

where n denotes the outward unit normal to Γ. Because of the Clarke subdifferential in (2.197), the problem will be formulated as a hemivariational inequality and then it will be embedded into a more general class of second-order evolution inclusions. Due to the multivalued term in the problem, the uniqueness of weak solutions is not expected.

Let $j : \Omega \times \mathbf{R} \to \mathbf{R}$ be defined as a minimum of two convex functions, that is, $j(x,s) = h(x) \min\{j_1(s), j_2(s)\}$ for $x \in \Omega$ and $s \in \mathbf{R}$, where $h \in L_\infty(\Omega)$, $j_1(s) = as^2$ and $j_2(s) = \frac{a}{2}(s^2 + 1)$ with $a > 0$. Then

$$\partial j(x,s) = h(x) \times \begin{cases} as & \text{if } s \in (-\infty, -1) \cup (1, +\infty) \\ 2as & \text{if } s \in (-1, 1) \\ [a, 2a] & \text{if } s = 1 \\ [-2a, -a] & \text{if } s = -1. \end{cases}$$

The model example can be modified to obtain nonmonotone zigzag relations which describe the adhesive contact laws for a granular material and a reinforced concrete, for example, the stick-slip law and the fiber bundle model law.

Another example which satisfies $H(j)$ is a superpotential of d.c. (difference of convex functions) type, that is, $j(s) = j_1(s) - j_2(s)$, where $j_1, j_2 : \mathbf{R} \to \mathbf{R}$ are convex functions.

We now turn to the variational formulation of the problem (2.189)–(2.198). We introduce the spaces for the displacement and electric potential:

$$V = \{v \in H^1(\Omega; \mathbf{R}^d) : v = 0 \text{ on } \Gamma_D\}, \tag{2.199}$$

$$\Phi = \{\varphi \in H^1(\Omega) : \varphi = 0 \text{ on } \Gamma_a\}$$

which are closed subspaces of $H^1(\Omega; \mathbf{R}^d)$ and $H^1(\Omega)$, respectively. Let $H = L_2(\Omega; \mathbf{R}^d)$ and $\mathscr{H} = L_2(\Omega; S_d)$ be Hilbert spaces equipped with the inner products $\langle u, v\rangle_H = \int_\Omega u \cdot v dx$, $\langle \sigma, \tau\rangle_{\mathscr{H}} = \int_\Omega \sigma : \tau dx$. Then the spaces (V, H, V^*) form an evolution triple of spaces. On V, we consider the inner product and the corresponding norm given by $\langle u, v\rangle_V = \langle \varepsilon(u), \varepsilon(v)\rangle_{\mathscr{H}}$, $\|v\|_V = \|\varepsilon(v)\|_{\mathscr{H}}$ for $u, v \in V$. From the Korn inequality $\|v\|_{H^1(\Omega;\mathbf{R}^d)} \le C\|\varepsilon(v)\|_{\mathscr{H}}$ for $v \in V$ with $C > 0$, it follows that $\|\cdot\|_{H^1(\Omega;\mathbf{R}^d)}$ and $\|\cdot\|_V$ are equivalent norms on V. Thus, $(V, \|\cdot\|_V)$ is a Hilbert space. On Φ, we consider the inner product $(\varphi, \psi)_\Phi = (\varphi, \psi)_{H^1(\Omega)}$ for $\varphi, \psi \in \Phi$. Then, $(\Phi, \|\cdot\|_\Phi)$ is also a Hilbert space.

Assuming sufficient regularity of the functions involved in the problem (2.189)–(2.198), multiplying (2.189) by $v \in V$ and using integration by parts, we have

$$\langle u''(t), v\rangle + \langle \sigma(u), \varepsilon(v)\rangle_{\mathscr{H}} - \int_{\Gamma} \sigma n \cdot v d\Gamma(x) = \langle f_1(t) + f_2(t), v\rangle - \langle \gamma u'(t), v\rangle$$

for a.e. $t > 0$. Since, by (2.194), we have $\int_{\Gamma} \sigma n \cdot v d\Gamma = \int_{\Gamma_N} g(t) \cdot v d\Gamma$ and (2.197) implies

$$-\int_{\Omega} f_2(x) \cdot v(x) dx \le \int_{\Omega} j^0(x, u(x,t); v(x)) dx,$$

we obtain

$$\langle u''(t), v\rangle + \langle \gamma u'(t), v\rangle + \langle \sigma(u), \varepsilon(v)\rangle_{\mathscr{H}} + \int_{\Omega} j^0(x, u(x,t); v(x)) dx \ge \langle F, v\rangle \tag{2.200}$$

where

$$\langle F, v\rangle = \langle f_1, v\rangle + \int_{\Gamma_N} g \cdot v d\Gamma \text{ for } v \in V.$$

Let $\psi \in \Phi$. From (2.190), again by using integration by parts and the condition (2.196), we have

$$-\langle D, \nabla\psi\rangle_H = 0. \tag{2.201}$$

Now inserting (2.191) into (2.200) and (2.192) into (2.201), we obtain the following variational formulation: for $-\infty < \tau < T < +\infty$, find $u \in C([\tau, T]; V) \cap C^1([\tau, T]; V)$ and $\varphi \in L_2(\tau, T; H)$ such that $u'' \in \mathscr{V}^*_{,\mathscr{T}}$, where $\mathscr{V}^*_{\tau,T} = L_2(\tau, T; V^*)$ and

$$\begin{cases} \langle u''(t), v\rangle + \langle \gamma u'(t), v\rangle + \langle \mathscr{A}\varepsilon(u), \varepsilon(v)\rangle_{\mathscr{H}} + \langle \mathscr{P}^T \nabla\varphi, \varepsilon(v)\rangle_{\mathscr{H}} + \\ + \int_{\Omega} j^0(x, u; v) dx \ge \langle F, v\rangle \quad \text{a.e. } t, \text{ for all } v \in V \\ \langle \mathscr{B}\nabla, \nabla\rangle_H = \langle \mathscr{P}\varepsilon(u), \nabla\psi\rangle_H \quad \text{a.e. } t, \text{ for all } \psi \in \Phi \\ u(0) = u_0, \ u'(0) = u_1. \end{cases} \tag{2.202}$$

We need the following hypotheses for the constitutive tensors.

$\underline{H(a)}$: The elasticity tensor field $a = (a_{ijkl})$ satisfies $a_{ijkl} \in L_\infty(\Omega)$, $a_{ijkl} = a_{klij}$, $a_{ijkl} = a_{jikl}$, $a_{ijkl} = a_{ijlk}$ and $a_{ijkl}(x)\tau_{ij}\tau_{kl} \ge \alpha\tau_{ij}\tau_{ij}$ for a.e $x \in \Omega$ and all $\tau = (\tau_{ij}) \in S_d$ with $\alpha > 0$.

$\underline{H(p)}$: The piezoelectric tensor field $p = (p_{ijk})$ satisfies $p_{ijk} = p_{ikj} \in L_\infty(\Omega)$.

$\underline{H(\beta)}$: The dielectric tensor field $\beta = (\beta_{ij})$ satisfies $\beta_{ij} = \beta_{ji} \in L_\infty(\Omega)$ and $\beta_{ij}(x)\xi_i\xi_j \ge m_\beta|\xi|^2_{\mathbf{R}^d}$ for a.e. $x \in \Omega$ and all $\xi = (\xi_i) \in \mathbf{R}^d$ with m_β¿0.

We define the following bilinear forms $a : V \times V \to \mathbf{R}$, $b : V \times \Phi \to \mathbf{R}$, $b^T : \Phi \times V \to \mathbf{R}$ and $c : \Phi \times \Phi \to \mathbf{R}$ by

$$a(u, v) = \int_\Omega a_{ijkl}(x) \frac{\partial u_k}{\partial x_l} \frac{\partial v_i}{\partial x_j} dx \quad \text{for} \quad u, v \in V,$$

$$b(u, \varphi) = \int_\Omega p_{ijk}(x) \frac{\partial u_i}{\partial x_j} \frac{\partial \varphi}{\partial x_k} dx \quad \text{for} \quad u \in V, \quad \varphi \in \Phi,$$

$$b^T(\varphi, u) = \int_\Omega p_{kij}(x) \frac{\partial \varphi}{\partial x_k} \frac{\partial u_i}{\partial x_j} dx \quad \text{for} \quad \varphi \in \Phi, \quad u \in V,$$

$$c(\varphi, \psi) = \int_\Omega \beta_{ij}(x) \frac{\partial \varphi}{\partial x_i} \frac{\partial \psi}{\partial x_j} dx \quad \text{for} \quad \varphi, \psi \in \Phi.$$

Then we have

$$\begin{aligned} a(u, v) &= \langle \mathscr{A}\varepsilon(u), \varepsilon(v)\rangle_{\mathscr{H}} \quad \text{for } u, v \in V, \\ b(u, \varphi) &= \langle \mathscr{P}\varepsilon(u), \nabla\varphi\rangle_H \quad \text{for } u \in V,\ \varphi \in \Phi, \\ b^T(\varphi, u) &= \langle \mathscr{P}\nabla\varphi, \varepsilon(u)\rangle_{\mathscr{H}} \quad \text{for } \varphi \in \Phi,\ u \in V, \\ c(\varphi, \psi) &= \langle \mathscr{B}\nabla\varphi, \nabla\psi\rangle_H \quad \text{for } \varphi, \psi \in \Phi, \end{aligned}$$

where the elasticity operator $\mathscr{A} : \Omega \times S_d \to S_d$ is given by $\mathscr{A}(x, \varepsilon) = a(x)\varepsilon$, $a(x) = (a_{ijkl}(x))$, the piezoelectric operator $\mathscr{P} : \Omega \times S_d \to \mathbf{R}^d$ is given by $\mathscr{P}(x, \varepsilon) = p(x)\varepsilon$, $p(x) = (p_{ijk}(x))$, the transpose to $\mathscr{P}$, $\mathscr{P}^T : \Omega \times \mathbf{R}^D \to S_d$ is given by $\mathscr{P}^T(x, \xi) = p^T(x)\xi$, $p^T(x) = (p^T_{ijk}(x)) = (p_{kij})$, and the electric permittivity operator $\mathscr{B} : \Omega \times \mathbf{R}^d \to \mathbf{R}^d$ is defined by $\mathscr{B}(x, \xi) = \beta(x)\xi$, $\beta(x) = (\beta_{ij(x)})$.

Using the above notation, the problem (2.202) is formulated as follows: find $u \in C([\tau, T]; V) \cap C^1([\tau, T]; V)$ and $\varphi \in L_2(\tau, T; H)$ such that $u'' \in \mathscr{V}_{\tau,T^*}$ and

$$\begin{cases} \langle u''(t), v\rangle + \langle \gamma u'(t), v\rangle + a(u(t), v) + b^T(\varphi(t), v) + \\ + \int_\Omega j^0(x, t, u; v)dx \geq \langle F, v\rangle \quad \text{a.e. } t, \quad \text{for all } v \in V \\ c(\varphi(t), \psi) = b(u(t), \psi) \quad \text{a.e. } t, \quad \text{for all } \psi \in \Phi \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{2.203}$$

The problem (2.203) is a system coupled with the hemivariational inequality for the displacement and a time-dependent stationary equation for the electric potential. We need now some auxiliary results and notation.

We remark that under hypotheses $H(p)$ and $H(\beta)$, for any $z \in V$, there exists a unique element $\varphi_z \in \Phi$ such that

$$c(\varphi_z, \psi) = b(z, \psi) \quad \text{for all} \quad \psi \in \Phi$$

and the map $C : V \to \Phi$ given by $Cz = \varphi_z$ is linear and continuous.

As a corollary, we obtain the following result: if $H(p)$, $H(\beta)$ hold and $u \in \mathscr{V}$, where $\mathscr{V} = L_2(\tau, T; V)$, then the problem

$$\begin{cases} \text{find } \varphi \in L_2(\tau, T; \Phi) \quad \text{such that} \\ c(\varphi(t), \psi) = b(u(t), \psi) \quad \text{for a.e. } t \in (\tau, T), \quad \text{for all } \psi \in \Phi \end{cases}$$

admits a unique solution $\varphi \in L_2(\tau, T; \Phi)$ and $\|\varphi\|_{L_2(\tau,T;\Phi)} \le c\|u\|_{\mathscr{V}_{\tau,T}}$ with $c > 0$. For a.e. $t \in (\tau, T)$, we have $\varphi(t) = Cu(t)$, where the operator C is defined in Lemma 3.1 of [23].

Next, since for every $\varphi \in \Phi$, the linear form $v \to b^T(\varphi, v)$ is continuous on V, so there exists a unique element $B\varphi \in V^*$ such that $b^T(\varphi, v) = \langle B\varphi, v\rangle_{V^* \times V}$ for all $v \in V$ and $B \in \mathscr{L}(\Phi, V^*)$. We observe that

$$b^T(\varphi, v) = \langle \mathscr{P}\nabla\varphi, \varepsilon(v)\rangle_{\mathscr{H}} = \int_\Omega \mathscr{P}^T\nabla\varphi : \varepsilon(v)dx \tag{2.204}$$

$$= \int_\Omega \mathscr{P}\varepsilon(v)\cdot\nabla\varphi dx = \langle \mathscr{P}\varepsilon(v), \nabla\varphi\rangle_H = b(v, \varphi) \quad \text{for all } v \in V, \quad \text{and } \varphi \in \Phi.$$

Analogously, we introduce the operator $A \in \mathscr{L}(V, V^*)$ such that $a(u, v) = \langle Au, v\rangle$ for all $u, v \in V$.

We are now in a position to reformulate the system (2.203). Since for a fixed $u \in \mathscr{V}$, the second equation in (2.203) is uniquely solvable (cf. Corollary 1 in [23]), we have

$$b^T(\varphi(t), v) = \langle B\varphi(t), v\rangle = \langle BCu(t), v\rangle \quad \text{for all } v \in V \quad \text{and a.e. } t \in (\tau, T).$$

Thus, the problem (2.203) takes the form: find $u \in C([\tau, T]; V) \cap C^1([\tau, T]; V)$ such that $u'' \in \mathscr{V}^*_{,\mathscr{T}}$ and

$$\begin{cases} \langle u''(t) + Ru'(t) + Gu(t), v\rangle + \int_\Omega j^0(x, u; v)dx \ge \langle F, v\rangle \\ \text{a.e. } t, \quad \text{for all } v \in V \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases} \tag{2.205}$$

where $R : H \to V^*$ and $G : V \to V^*$ are given by $Rv = \gamma v$ for $v \in H$ and $Gv = Av + BCv$ for $v \in V$, respectively.

The existence of solutions to the hemivariational inequality (2.205) will be a consequence of a more general result provided in [23]. We remark that operators R and G satisfy such properties: if $\gamma \in L_\infty(\Omega)$, $\gamma \ge \gamma_0 > 0$, then the operator $R : H \to H$ defined by $Rv = \gamma v$ is linear continuous symmetric and coercive. Under the hypotheses $H(a)$, $H(p)$, and $H(\beta)$, the operator $G : V \to V^*$ defined by $G = A + BC$ is linear, bounded, symmetric, and coercive.

Finally, we obtain the following second-order evolution inclusion: find $u \in C([\tau, T]; V) \cap C^1([\tau, T]; V)$ such that $u'' \in \mathscr{V}^*_{,\mathscr{T}}$ and

$$\begin{cases} u''(t) + Ru'(t) + Gu(t) + \partial J(t, u(t)) \ni F(t) \quad \text{a.e.} \quad t \in (0, T) \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{2.206}$$

We need the following hypotheses:

$\underline{H(R)}$ $R : H \to H$ is a linear symmetric such that $\exists \gamma > 0 : (Rv, v)_H = \gamma \|v\|_H^2$ $\forall v \in H$.

$\underline{H(G)}$ $G : V \to V^*$ is linear and symmetric and $\exists c_G > 0 : \langle Gv, v\rangle_V \geq c_G \|v\|_V^2$ $\forall v \in V$.

$\underline{H(J)}$ $J : H \to \mathbf{R}$ is a function such that:

(i) $J(\cdot)$ is locally Lipschitz and regular one [12].

(ii) $\exists c_1 > 0$: $\|\partial J(v)\|_+ \leq c_1(1 + \|v\|_H)$ $\forall v \in H$.

(iii) $\exists c_2 > 0$:

$$[\partial J(v), v]_- \geq -\lambda \|v\|_H^2 - c_2 \quad \forall v \in H,$$

where $\partial J(v) = \{p \in H \,|\, (p, w)_H \leq J^\circ(v; w)\ \forall w \in H\}$ denotes the Clarke subdifferential of $J(\cdot)$ at a point $v \in H$ (see [12] for details), $\lambda \in (0, \lambda_1)$, $\lambda_1 > 0$: $c_G \|v\|_V^2 \geq \lambda_1 \|v\|_H^2$ $\forall v \in V$.

$\underline{(H_0)}$ V is a Hilbert space.

We remark that Condition $H(j)$(iii) is technical condition provides only dissipation of multivalued (in general case) dynamical system constructed on all weak solutions of Problem (2.189). This condition is not connected with the nonsmoothness of j.

In [23], it is proved that if hypotheses $H(R)$, $H(G)$, $H(J)$, and (H_0) hold, then the problem (2.206) has a solution. Due to the previous results, we can investigate a long-time behavior of all weak solutions of the problem (2.206) under similar but some stronger (providing a dissipation) conditions. In particular, we study the structure of the global and trajectory attractors.

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Chapter 3
Attractors for Lattice Dynamical Systems

A lot of processes coming from Physics, Chemistry, Biology, Economy, and other sciences can be described using systems of reaction-diffusion equations. In this chapter, we study the asymptotic behavior of the solutions of a system of infinite ordinary differential equations (a lattice dynamical system) obtained after the spacial discretization of a system of reaction-diffusion equations in an unbounded domain. This kind of dynamical systems is then of importance in the numerical approximations of physical problems.

Lattice dynamical systems appear also in practice as discrete models, for example, in cellular neural networks [10] or in chemical reaction systems and biology (see [4, 12, 14]) among others (see [31] for more references).

Although the study in lattice dynamical systems of properties like the chaotic behavior of solutions, the stability of equilibrium points, bifurcations or the existence of travelling waves was very rich in the 1990s (see [9] and the references therein), the study of the global asymptotic behavior of solutions as an infinite-dimensional dynamical system (in particular, the study of the global attractor) is more recent. The existence of global attractors for lattice dynamical systems has been studied by many authors in the last years (see, e.g., [1, 3, 5, 18, 20, 21, 27–34]). These works contain discretizations (in unbounded domains) of reaction-diffusion systems, hyperbolic systems, the Schrödinger equation, and others. Also, in [34], the problem of estimation of the fractal dimension of the global attractor is considered.

In these works, the authors impose conditions ensuring the uniqueness of the Cauchy problem like Lipschitz or differentiability conditions on the nonlinear term. In this way, a semigroup of operators is defined in an appropriate one-parameter phase space (usually, an l^p space or a weighted l^p space), proving then the existence of a compact global attractor. Moreover, the upper semicontinuity of the attractor with respect to finite-dimensional approximations is shown in most of the chapters.

In this chapter, we study the asymptotic behavior of the lattice dynamical system generated by the discretization of the system reaction-diffusion

M.Z. Zgurovsky et al., *Evolution Inclusions and Variation Inequalities for Earth Data Processing III*, Advances in Mechanics and Mathematics 27,
DOI 10.1007/978-3-642-28512-7_3,

$$\begin{cases} \frac{\partial u}{\partial t} = \widetilde{a}\Delta u - f(x,u), x \in \mathbf{R}, t > 0, \\ u(0) = u^0 \in (L_2(\mathbf{R}))^m, \end{cases} \tag{3.1}$$

given by

$$\begin{cases} \dot{u}_{j,i} = \sum_{k=1}^{m} a_{jk} \left(u_{k,i-1} - 2u_{k,i} + u_{k,i+1}\right) - f_{j,i}\left(u_{\cdot,i}\right), \\ u_{j,i}(0) = \left(u^0\right)_{j,i}, \end{cases} \tag{3.2}$$

where $a = \frac{1}{h^2}\widetilde{a}$ is a real matrix of order m and the parameter h is the step of the spacial discretization.

We note that the existence of a global attractor of system (3.1) in the phase space $\left(L^2(\mathbf{R})\right)^m$ under appropriate conditions (not ensuring the uniqueness of the Cauchy problem) is given in [22]. Now, assuming that a is a real matrix with nonnegative symmetric part and that

$$f_{\cdot,i}\left(u_{\cdot,i}\right) = \left(f_{1,i}\left(u_{1,i},\ldots,u_{m,i}\right),\ldots,f_{m,i}\left(u_{1,i},\ldots,u_{m,i}\right)\right),\ i \in \mathbf{Z},$$

are continuous functions verifying appropriate dissipative and growth conditions, we prove the existence of a global compact attractor for system (3.2). Hence, this attractor can be considered as an approximation of the attractor of system (3.1).

Comparing with the previous papers mentioned above, we do not impose conditions ensuring the uniqueness of the Cauchy problem, as the function f is not assumed to be locally Lipschitz (in fact, we give an example of a system in which at least two solutions corresponding to an initial data exist). This allows to consider a larger class of systems of type (3.2). As it was already pointed out in the first chapter that the study of the asymptotic behavior of differential equations without uniqueness of solutions has been developed intensively in the last years and is motivated by important equations of the mathematical physics like the 3D Navier–Stokes system (see [2, 8, 15, 17, 24, 25]) or the Ginzburg–Landau system (see [7, 16]).

We prove first a Peano's theorem for system (3.2). After that, we define a multi-valued semiflow (instead of a semigroup of operators) and prove the existence of a global attractor for it. Finally, we study the upper semicontinuity of the attractor with respect to finite-dimensional approximations. These results were proved in [23].

3.1 Existence of Solutions

Consider the following first-order lattice dynamical system

$$\begin{cases} \dot{u}_{j,i} = \sum_{k=1}^{m} a_{jk} \left(u_{k,i-1} - 2u_{k,i} + u_{k,i+1}\right) - f_{j,i}\left(u_{\cdot,i}\right), \\ u_{j,i}(0) = \left(u^0\right)_{j,i}, \end{cases} \tag{3.3}$$

where $a = \left(a_{jk}\right)$ is a real matrix and

$$f_{\cdot,i}\left(u_{\cdot,i}\right) = \left(f_{1,i}\left(u_{1,i}, \ldots, u_{m,i}\right), \ldots, f_{m,i}\left(u_{1,i}, \ldots, u_{m,i}\right)\right), i \in \mathbf{Z}.$$

Here, for a fixed $i \in \mathbf{Z}$, $u_{\cdot,i}$, $f_{\cdot,i}$ refer to the vectors $(u_{1,i}, \ldots, u_{m,i})$, $(f_{1,i}, \ldots, f_{m,i}) \in \mathbf{R}^m$, respectively.

Throughout this chapter, we denote $l_m^2 := l^2 \times l^2 \times \overset{(m)}{\cdots} \times l^2$, which is a Hilbert space endowed with the scalar product

$$(u, v)_{l_m^2} := \sum_{j=1}^{m} \sum_{i \in \mathbf{Z}} u_{j,i} v_{j,i}, u, v \in l_m^2,$$

and the norm $\|u\|_{l_m^2} := \sum_{i \in \mathbf{Z}} \sum_{j=1}^{m} \left|u_{j,i}\right|^2 = \sum_{i \in \mathbf{Z}} |u_{\cdot,i}|^2_{\mathbf{R}^m}$. Also, we shall use the scalar products $(x, y)_{\mathbf{R}^m} := \sum_{j=1}^{m} x_j y_j,\ x, y \in \mathbf{R}^m$, $(u, v)_{l^2} := \sum_{i \in \mathbf{Z}} u_i v_i$, $u, v \in l^2$.

As before, for $u \in l_m^2$ and a fixed $1 \leq j \leq m$, the notation $u_{j,\cdot}$ is used for the element $\left(\ldots, u_{j,-1}, u_{j,0}, u_{j,1}, u_{j,2}, \ldots\right) \in l^2$.

We shall assume the following conditions:

(H1) For all $x \in \mathbf{R}^m,\ i \in \mathbf{Z}$,

$$\left(f_{\cdot,i}(x), x\right)_{\mathbf{R}^m} \geq \alpha\,|x|^2_{\mathbf{R}^m} - c_{0,i},$$

where $c_0 \in l^1, \alpha > 0$.

(H2) For all $x \in \mathbf{R}^m,\ i \in \mathbf{Z}$,

$$|f_{\cdot,i}(x)|_{\mathbf{R}^m} \leq C\left(|x|_{\mathbf{R}^m}\right)|x|_{\mathbf{R}^m} + c_{1,i},$$

where $c_1 \in l^2$, $c_{1,i} \geq 0$, and $C(\cdot) \geq 0$ is a continuous increasing function.

(H3) The real matrix $a = (a_{k,l})$ has a nonnegative symmetric part, that is, there exists $\beta \geq 0$ such that $\frac{1}{2}(a + a^t) \geq \beta I_m$.

(H4) The maps $f_{j,i} : \mathbf{R}^m \to \mathbf{R}$ are continuous.

Observe that from (H3), we deduce $(ax, x)_{\mathbf{R}^m} = \left(\frac{1}{2}(a + a^t)\,x, x\right)_{\mathbf{R}^m} \geq \beta\,\|x\|^2_{\mathbf{R}^m}$.

We shall rewrite now system (3.3) in a matricial form.

We define the operator $A : l_m^2 \to l_m^2$ by $(Au)_{j,i} := -u_{j,i-1} + 2u_{j,i} - u_{j,i+1},\ i \in \mathbf{Z}$. It is easy to see that the operator A is equal to $\left[\hat{A}\ \ldots\ \hat{A}\right]$, where $\hat{A} : l^2 \to l^2$ is defined by

$$\left(\hat{A}v\right)_i = -v_{i-1} + 2v_i - v_{i+1}, \text{ for } v \in l^2.$$

Hence, for $u = (u_{1,\cdot}, \ldots, u_{m,\cdot}) \in l_m^2$, we have

$$Au := \left[\hat{A}u_{1,\cdot}\ \ldots\ \hat{A}u_{m,\cdot}\right].$$

Also, we define the operators $\hat{B}, \hat{B}^* : l^2 \to l^2$ by

$$\left(\hat{B}v\right)_i := v_{i+1} - v_i, \; \left(\hat{B}^*v\right)_i := v_{i-1} - v_i .$$

It is easy to check that

$$\hat{A} = \hat{B}^*\hat{B} = \hat{B}\hat{B}^*,$$
$$(B^*u, v)_{l_m^2} = (u, Bv)_{l_m^2} ,$$

where $B := \left[\hat{B} \; \dots \; \hat{B}\right]$ and $B^* := \left[\hat{B}^* \; \dots \; \hat{B}^*\right]$. Moreover,

$$(aAu, v)_{l_m^2} = (aBu, Bv)_{l_m^2} .$$

This property is proved as follows:

$$\begin{aligned}(aAu, v)_{l_m^2} &= \textstyle\sum_{j=1}^m \left(a_{1j}\hat{A}u_{j,\cdot}, v_{1,\cdot}\right)_{l^2} + \overset{(m)}{\cdots} + \sum_{j=1}^m \left(a_{mj}\hat{A}u_{j,\cdot}, v_{m,\cdot}\right)_{l^2} \\ &= \textstyle\sum_{j=1}^m \left(a_{1j}\hat{B}u_{j,\cdot}, \hat{B}v_{1,\cdot}\right)_{l^2} + \overset{(m)}{\cdots} + \sum_{j=1}^m \left(a_{mj}\hat{B}u_{j,\cdot}, \hat{B}v_{m,\cdot}\right)_{l^2} \\ &= (aBu, Bv)_{l_m^2} .\end{aligned}$$

Remark 3.1. This property is true for any real matrix a.

We need to prove that the operator $\hat{f}: l_m^2 \longrightarrow l_m^2$ given by $\hat{f}(u) \equiv [f_{1,\cdot}(u), \dots, f_{m,\cdot}(u)]$ is well defined, that is, $\hat{f}(u) \in l_m^2$, $\forall u \in l_m^2$. Also, we check the same property for the map $F : l_m^2 \to l_m^2$ defined as

$$[F(u)]_{j,i} := -(aAu)_{j,i} - f_{j,i}(u_{\cdot,i}), \; j = 1, \dots, m, \; i \in \mathbf{Z}.$$

Indeed,

$$\left|[F(u)]_{j,i}\right| \le \left[\sum_{k=1}^m \left|a_{jk}\right| \left|-u_{k,i-1} + 2u_{k,i} - u_{k,i+1}\right| + \left|f_{j,i}(u_{\cdot,i})\right|\right], \tag{3.4}$$

and then,

$$\left|[F(u)]_{j,i}\right|^2 \le 2m\left[\sum_{k=1}^m \left|a_{jk}\right|^2 \left|-u_{k,i-1} + 2u_{k,i} - u_{k,i+1}\right|^2\right] + 2\left|f_{j,i}(u_{\cdot,i})\right|^2 .$$

By (H2), we have

$$|f_{\cdot,i}(u_{\cdot,i})|^2_{\mathbf{R}^m} \le 2\left[\eta^2(u)\,|u_{\cdot,i}|^2_{\mathbf{R}^m} + c_{1,i}^2\right], \tag{3.5}$$

$$\left|[F(u)]_{\cdot,i}\right|^2_{\mathbf{R}^m} \le 2m^2\widetilde{a}^2\left[\sum_{k=1}^m \left|-u_{k,i-1}+2u_{k,i}-u_{k,i+1}\right|^2\right] \\ +2\left[\eta(u)\left|u_{\cdot,i}\right|_{\mathbf{R}^m}+c_{1,i}\right]^2, \tag{3.6}$$

where $\widetilde{a} := \max_{1\le j,k\le m}\left|a_{jk}\right|$, $\eta(u) := \max_{i\in\mathbf{Z}} C\left(\left|u_{\cdot,i}\right|_{\mathbf{R}^m}\right)$. Note that $\eta(u)$ exists and is finite for any $u \in l^2_m$. Also, $\eta(u)$ is bounded if u belongs to a bounded set of l^2_m.

It follows from $u \in l^2_m$, $c_1 \in l^2$ and (3.5)–(3.6) that $\hat{f}(u)$, $F(u) \in l^2_m$.

On the other hand, we can obtain by using (H2) and (3.4) the following estimate of the norm of $F(u)$:

$$\|F(u)\|^2_{l^2_m} \le 2m\sum_{j=1}^m\left[\sum_{k=1}^m\left|a_{j,k}\right|^2\left(\sum_{i\in\mathbf{Z}}\left(-u_{k,i-1}+2u_{k,i}-u_{k,i+1}\right)^2\right)\right] \\ +2\sum_{i\in\mathbf{Z}}\left(c_{1,i}+\eta(u)\left|u_{\cdot,i}\right|_{\mathbf{R}^m}\right)^2 \tag{3.7} \\ \le 2\widetilde{a}^2K_1\|u\|^2_{l^2_m}+K_2\|c_1\|^2_{l^2}+K_3\eta(u)\|u\|^2_{l^2_m} := K(a,m,c_1,u).$$

Thus, (3.3) is rewritten as

$$\begin{cases} \dot{u} = -aAu - \widehat{f}(u) = F(u), \\ u(0) = u^0 \in l^2_m. \end{cases} \tag{3.8}$$

Now, we shall obtain that the map F is continuous. As the operator $aA : l^2_m \to l^2_m$ is obviously continuous, for this aim, it is sufficient to check the continuity of the nonlinear map $\hat{f}$.

Proposition 3.1. *Assume that* $(H2)$, $(H4)$ *hold. Let* $\{u^n\}_{n\in\mathbf{N}} \subset l^2_m$ *be such that* $u^n \to u^0$ *in* l^2_m. *Then* $\hat{f}(u^n) \to \hat{f}\left(u^0\right)$ *in* l^2_m. *Hence,* $\hat{f}$, F *are continuous maps.*

Proof. Since $u^n \to u^0$ in l^2_m, $u^0 \in l^2_m$, $c_1 \in l^2$, it is clear that for any $\epsilon > 0$, there exist $n_0(\epsilon)$ and $k_0(\epsilon)$ such that

$$\left\|u^n-u^0\right\|_{l^2_m} < \frac{\epsilon}{2},\ \sum_{j=1}^m\sum_{|i|>k_0}\left|u^0_{j,i}\right|^2 < \frac{\epsilon}{2},\ \sum_{|i|\ge k_0}\left|c_{1,i}\right|^2 \le \epsilon,$$

if $n > n_0$. Thus,

$$\sum_{j=1}^m\sum_{|i|>k_0}\left|u^n_{j,i}\right|^2 \le 2\sum_{j=1}^m\sum_{|i|>k_0}\left[\left|u^n_{j,i}-u^0_{j,i}\right|^2+\left|u^0_i\right|^2\right] < 2\epsilon,\ \forall n \ge n_0.$$

Also, by (H4), we can choose $n_1(\epsilon, k_0) \geq n_0$ such that if $n \geq n_1$, then

$$\sum_{j=1}^{m} \sum_{|i|\leq k_0} \left| f_{j,i}\left(u_{\cdot,i}^{n}\right) - f_{j,i}\left(u_{\cdot,i}^{0}\right)\right|^2 \leq \epsilon.$$

Therefore, using (H2), we obtain

$$\begin{aligned}
\left\| \hat{f}(u^n) - \hat{f}\left(u^0\right)\right\|_{l_m^2}^2 &= \sum_{j=1}^{m} \sum_{|i|\leq k_0} \left| f_{j,i}\left(u_{\cdot,i}^{n}\right) - f_{j,i}\left(u_{\cdot,i}^{0}\right)\right|^2 \\
&\quad + \sum_{j=1}^{m} \sum_{|i|> k_0} \left| f_{j,i}\left(u_{\cdot,i}^{n}\right) - f_{j,i}\left(u_{\cdot,i}^{0}\right)\right|^2 \\
&\leq \epsilon + 4 \sum_{|i|>k_0} \left(|c_{1,i}|^2 + \eta^2 \left|u_{\cdot,i}^{n}\right|_{\mathbf{R}^m}^2 \right) \\
&\quad + 4 \sum_{|i|>k_0} \left(|c_{1,i}|^2 + \eta^2 \left|u_{\cdot,i}^{0}\right|_{\mathbf{R}^m}^2 \right) \leq k\epsilon,
\end{aligned}$$

where $\eta = \max\{\max_{i\in\mathbf{Z}} C\left(\left|u_{\cdot,i}^{0}\right|_{\mathbf{R}^m}\right), \sup_{i\in\mathbf{Z},n\in\mathbf{N}} C\left(\left|u_{\cdot,i}^{n}\right|_{\mathbf{R}^m}\right)\}$ is a finite nonnegative number. □

We shall prove further that the map $F : l_m^2 \to l_m^2$ is weakly continuous. For this aim, it is enough to check the weakly continuity of the map $\widehat{f} : l_m^2 \to l_m^2$.

Lemma 3.1. *Assume that (H2), (H4) hold. Then, the map $\widehat{f} : l_m^2 \to l_m^2$ is weakly continuous.*

Proof. We note that as the space l_m^2 endowed with the weak topology satisfies the first axiom of countability, it is not difficult to show that f is weakly continuous if and only if $u_n \to u_0$ weakly in l_m^2 implies $f(u_n) \to f(u_0)$.

Take an arbitrary $\xi \in l^2$. Then for any $\epsilon > 0$, there exists $K_1(\epsilon, \xi)$ such that $\sum_{|i|\geq K_1} |\xi_{\cdot,i}|^2 \leq \epsilon$. Due to condition (H2), we have

$$\left\| \widehat{f}(u)\right\|_{l_m^2}^2 \leq \sum_{i\in\mathbf{Z}} \left(c_{1,i} + C\left(|u_{\cdot,i}|_{\mathbf{R}^m}\right) |u_{\cdot,i}\|_{\mathbf{R}^m}\right)^2 \leq 2\left(\|c_1\|_{l^2}^2 + \eta^2(u)\|u\|_{l_m^2}^2\right), \tag{3.9}$$

where $\eta(u) = \max_{i\in\mathbf{Z}} C\left(|u_{\cdot,i}|_{\mathbf{R}^m}\right)$ Hence, if $\|u\|_{l_m^2} \leq R$, then there exists $M = M(R)$ such that $\left\|\widehat{f}(u)\right\|_{l_m^2} \leq M$. Thus, if $u^n \to u$ weakly in l_m^2, we have

$$\left|\left(\widehat{f}\left(u^{n}\right)-\widehat{f}(u),\xi\right)_{l_m^2}\right| \leq \sum_{|i|\leq K_1}\left|\left(f_{\cdot,i}\left(u^{n}_{\cdot,i}\right)-f_{\cdot,i}\left(u_{\cdot,i}\right)\right)\right|_{\mathbf{R}^m}\left|\xi_{\cdot,i}\right|_{\mathbf{R}^m}$$
$$+\left(\left\|\widehat{f}(u)\right\|_{l_m^2}+\left\|\widehat{f}\left(u^{n}\right)\right\|_{l_m^2}\right)\left(\sum_{|i|\geq K_1}\left|\xi_{\cdot,i}\right|^2_{\mathbf{R}^m}\right)^{\frac{1}{2}}$$
$$\leq \|\xi\|_{l_m^2}\left(\sum_{|i|\leq K_1}\left|\left(f_{\cdot,i}\left(u^{n}_{\cdot,i}\right)-f_{\cdot,i}\left(u_{\cdot,i}\right)\right)\right|^2_{\mathbf{R}^m}\right)^{\frac{1}{2}}+2M\epsilon,$$

for some M. For any $\varepsilon>0$, we choose $\epsilon=\frac{\varepsilon}{4M}$. Since $u_i^n\to u_i$ for all i and f_i are continuous, we obtain the existence of $N_1(\varepsilon,\xi)$ such that

$$\left(\sum_{|i|\leq K_1}\left|\left(f_{\cdot,i}\left(u^{n}_{\cdot,i}\right)-f_{\cdot,i}\left(u_{\cdot,i}\right)\right)\right|^2_{\mathbf{R}^m}\right)^{\frac{1}{2}}\leq\frac{\varepsilon}{2\|\xi\|_{l_m^2}}$$

(for $\xi\neq0$). Hence,

$$\left|\left(\widehat{f}\left(u^{n}\right)-\widehat{f}(u),\xi\right)_{l_m^2}\right|\leq\varepsilon \text{ if } n\geq N_1. \qquad\square$$

Now, we shall introduce a new version of Tartaglia's triangle, which will be useful in the proof of the existence of local solutions of system (3.3).

By $\begin{bmatrix}p\\k\end{bmatrix}$, we denote the elements of the triangle

$p=1$			2	4		
$p=2$		2	6	4		
$p=3$	2	8	10	4		
$\vdots$	2	10	18	10	4	
	2	12	28	32	18	4

where for every row p, the index k goes from 0 to p. This triangle is defined by the rule:

$$\begin{bmatrix}p\\0\end{bmatrix}=2,\ \begin{bmatrix}p\\p\end{bmatrix}=4,\ \begin{bmatrix}p+1\\k\end{bmatrix}=\begin{bmatrix}p\\k-1\end{bmatrix}+\begin{bmatrix}p\\k\end{bmatrix},\ \text{for all } 1\leq k\leq p-1. \tag{3.10}$$

Also, we can check that for $1\leq k\leq p$,

$$\begin{bmatrix}p\\k\end{bmatrix}\leq 4\binom{p}{k}. \tag{3.11}$$

Indeed, we proceed by induction. The case $p = 1$ is obvious, as $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4 \leq 4\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 4$. Assume that it is true for p.

For $p+1$ and $k = 1$, we can see that

$$\begin{bmatrix} p+1 \\ 1 \end{bmatrix} = 2p + 4 \leq 4p + 4 = 4\,(p+1) = 4\begin{pmatrix} p+1 \\ 1 \end{pmatrix},$$

where we have used the formula $\begin{bmatrix} q \\ 1 \end{bmatrix} = 4 + 2\,(q-1)$, for all $q \geq 1$.

If $k = p+1$, then $\begin{bmatrix} p+1 \\ p+1 \end{bmatrix} = 4 = 4\begin{pmatrix} p+1 \\ p+1 \end{pmatrix}$. Finally, for $2 \leq k < p+1$, we have by the hypothesis of induction that

$$\begin{bmatrix} p+1 \\ k \end{bmatrix} = \begin{bmatrix} p \\ k-1 \end{bmatrix} + \begin{bmatrix} p \\ k \end{bmatrix} \leq 4\left[\begin{pmatrix} p \\ k-1 \end{pmatrix} + \begin{pmatrix} p \\ k \end{pmatrix}\right] = 4\begin{pmatrix} p+1 \\ k \end{pmatrix}.$$

On the other hand, we need also the following estimate

$$\begin{pmatrix} n \\ k \end{pmatrix} \leq n^k, \tag{3.12}$$

which is straightforward to prove.

In the sequel, we shall write $B_{l_m^2}(u_0, R)$ for an open ball in l_m^2 centered at u_0 with radius R.

In the next theorem, we prove the existence of a local solution for (3.3). We note that in view of (3.7) and the fact that $\eta\,(u)$ is a bounded map, it follows that F is a bounded map in l_m^2.

As in infinite-dimensional Banach spaces the continuity of the map F is not enough in order to prove the existence of solutions [13], additional conditions are necessary. For example, in [6, 26], compacity conditions on F are used, whereas in [19], the author assumes that F is weakly continuous. Using the result in [19] and the weak continuity of F, one can obtain the existence of solutions. However, we will give an specific proof of Peano's theorem in our case.

Remark 3.2. Let $E := \max\{m \max_{1\leq k,l\leq m}\{|a_{kl}|\}, 1\}$. Then it is easy to see that for any $v \in \mathbf{R}^m$, it holds $\sum_{j=1}^m \left|a_{\cdot,j} v_j\right|_{\mathbf{R}^m} \leq E\,|v|_{\mathbf{R}^m}$. In particular, $\sum_{j=1}^m \left|a_{\cdot,j}\, y_{j,i}^0\,(t)\right|_{\mathbf{R}^m} \leq E\left|y_{\cdot,i}^0\,(t)\right|_{\mathbf{R}^m}$, for $y_{\cdot,i}^0\,(t) \in \mathbf{R}^m$.

Theorem 3.1. *Suppose that* $(H2)$, $(H4)$ *hold. Let* M *be a bound of the norm of* $F\,(u)$ *for* $u \in D = B_{l_m^2}\left(y^0, b\right)$, *with* $b > 0$. *Then there exists a solution* $y\,(\cdot) \in C\left(I, l_m^2\right)$ *of (3.3) defined in* $I := [0, t_0 + \hat{\alpha}]$, *where* $\hat{\alpha} < \min\{\frac{b}{M}, 1\}$ *is such that*

$$0 < 2\hat{\alpha}\,(1 + 4E + \eta) < 1, \tag{3.13}$$

$$\eta = \sup_{u \in B_{l_m^2}\left(y^0, b\right)} \max_{i \in \mathbf{Z}} C\left(|u_{\cdot,i}|_{\mathbf{R}^m}\right), \tag{3.14}$$

and E has been defined in Remark 3.2.

Remark 3.3. In this theorem, Conditions $(H1)$ and $(H3)$ are not necessary.

Proof. Let $\delta > 0$. We choose some function $y^0(\cdot) \in C\left([-\delta, 0], l_m^2\right)$ satisfying $y^0(0) = y^0$ and

$$\left\| y^0(t) - y^0 \right\|_{l_m^2} \leq b, \;\forall\, t \in [-\delta, 0]. \tag{3.15}$$

Let $\epsilon, \delta_\epsilon$ be such that $0 < \epsilon \leq \delta_\epsilon,\; \delta_\epsilon \leq \delta$. We define now the functions $y^\epsilon(\cdot) : [-\delta_\epsilon, \alpha_1] \to l_m^2$, where $\alpha_1 := \min\{\epsilon, \hat{\alpha}\}$, in the following way:

$$y^\epsilon(t) := \begin{cases} y^0(t), \text{ if } t \in [-\delta_\epsilon, 0], \\ y^0 + \int_0^t F\left(y^\epsilon(s - \epsilon)\right) ds, \text{ if } t \in [0, \alpha_1]. \end{cases} \tag{3.16}$$

It is clear that $y^\epsilon(\cdot) \in C\left([-\delta_\epsilon, \alpha_1]\,; l_m^2\right)$. Also, we have

$$\left\| y^\epsilon(t) - y^0 \right\|_{l_m^2} = \left\| \int_0^t F\left(y_0(r - \epsilon)\right) dr \right\|_{l_m^2} \leq M\,|t| \leq M\hat{\alpha} \leq M\,\frac{b}{M} = b, \tag{3.17}$$

$$\left\| y^\epsilon(t) - y^\epsilon(s) \right\|_{l_m^2} = \left\| \int_s^t F\left(y_0(r - \epsilon)\right) \right\|_{l_m^2} dr \leq M\,|t - s|\,, \tag{3.18}$$

for $t, s \geq 0$. We can extend y^ϵ in the intervals $[-\delta_\epsilon, \alpha_2]$, $[-\delta_\epsilon, \alpha_3]$, and so on, where $\alpha_i = \min\{\hat{\alpha}, i\epsilon\}$, in the same way as in (3.16). Hence, we can define y^ϵ in $[-\delta_\epsilon, \hat{\alpha}]$ in such a way that (3.17)–(3.18) hold and $y^\epsilon(t) \in C\left([-\delta_\epsilon, \hat{\alpha}]\,; l_m^2\right)$.

Let us consider the family of functions $\mathscr{F} = \{y^\epsilon(\cdot) : 0 < \epsilon \leq \delta\} \subset C\left([0, \hat{\alpha}]\,; l_m^2\right)$. We can see from (3.18) that it is an equicontinuous family.

Let $\epsilon_n \to 0$ be a sequence of real positive numbers such that $\frac{\hat{\alpha}}{\epsilon_n} = m_n \in \mathbf{N}$, and let $\delta_{\epsilon_n} = \epsilon_n$. Then we shall prove that for any $t^* \in [0, \hat{\alpha}]$, the sequence $\{y^{\epsilon_n}(t^*)\}$ is precompact. We can assume that $t^* > 0$, as the case $t^* = 0$ is trivial.

We need to check the existence of $w \in l_m^2$ and a subsequence $\{\epsilon_{n_k}\}$ such that for any $\xi > 0$, there exists $N(\xi)$ for which

$$\left\| y^{\epsilon_{n_k}}\left(t^*\right) - w \right\|_{l_m^2} \leq \xi, \text{ if } n \geq N.$$

First, from the estimate

$$\left\| y^\epsilon\left(t^*\right) \right\|_{l_m^2} \leq \left\| y^0 \right\|_{l_m^2} + M\,|t^*|$$

we obtain that $\{y_{\epsilon_n}(t^*)\}$ has a subsequence converging to some w weakly in l_m^2. We need to prove that the convergence is strong.

For this aim, we define the sequence $\{\xi_i^n\}_{i\in\mathbf{Z}}$ as follows:

$$\xi_i^n = \max\left\{\max_{t\in[-\delta_{\epsilon_n},0]}\{|y^0_{\cdot,i-1}(t)|_{\mathbf{R}^m}, |y^0_{\cdot,i}(t)|_{\mathbf{R}^m}, |y^0_{\cdot,i+1}(t)|_{\mathbf{R}^m}, |y^0_{\cdot,i}(t)|_{\mathbf{R}^m}\}, c_{1i}\right\}.$$

We shall need the following technical lemma:

Lemma 3.2. *For* $1 \le \hat{n} \le m_n$, *we define*

$$T_{p,i}^{n,\hat{n}} := (\epsilon_n E)^p \left\{\binom{p}{0}\xi^n_{i-p} + \binom{p}{1}\xi^n_{i-p+2} + \ldots + \binom{p}{p}\xi^n_{i+p}\right\}\binom{\hat{n}-1}{p}$$
$$\cdot\left[\begin{array}{c}\binom{\hat{n}-p}{0} \\ +\epsilon_n\left\{\binom{\hat{n}-p}{1} + \left[\begin{matrix}\hat{n}-p\\1\end{matrix}\right]E + \binom{\hat{n}-p}{1}\eta\right\} \\ +\epsilon_n^2\left\{\binom{\hat{n}-p}{2} + \left[\begin{matrix}\hat{n}-p\\2\end{matrix}\right]E + \binom{\hat{n}-p}{2}\eta\right\}(2E+\eta) \\ +\ldots \\ +\epsilon_n^{\hat{n}-p}\left\{\binom{\hat{n}-p}{\hat{n}-p} + \left[\begin{matrix}\hat{n}-p\\\hat{n}-p\end{matrix}\right]E + \binom{\hat{n}-p}{\hat{n}-p}\eta\right\}(2E+\eta)^{\hat{n}-p-1}\end{array}\right],$$

for $0 \le p \le \widehat{n}-1$. *Let* $r \in [(\hat{n}-1)\epsilon_n, \hat{n}\epsilon_n]$. *Then it holds*

$$|y^{\epsilon_n}_{\cdot,i}(r)|_{\mathbf{R}^m} \le \sum_{p=0}^{\hat{n}-1} T_{p,i}^{n,\hat{n}} := \mathbf{B}^n_{\hat{n},i}. \tag{3.19}$$

Proof. We proceed by induction on $\hat{n}$.

- Case $\hat{n}=1$.

  Let $r \in [0,\epsilon_n]$. Then

$$\begin{aligned}|y^{\epsilon_n}_{\cdot,i}(r)|_{\mathbf{R}^m} &= \left|y^0_{\cdot,i} + \int_0^r F_{\cdot,i}\left(y^{\epsilon_n}_{\cdot,i}(s-\epsilon)\right)ds\right|_{\mathbf{R}^m} \\ &\le |y^0_{\cdot,i}|_{\mathbf{R}^m} + \int_0^r\left[\sum_{j=1}^m\left|a_{\cdot,j}\left(y^{\epsilon_n}_{j,i-1} + 2y^{\epsilon_n}_{j,i} + y^{\epsilon_n}_{j,i+1}\right)(s-\epsilon)\right|_{\mathbf{R}^m} + \right. \\ &\quad \left. + \left|f_{\cdot,i}\left(y^{\epsilon_n}_{\cdot,i}(s-\epsilon)\right)\right|_{\mathbf{R}^m}\right]ds.\end{aligned}$$

Using condition (H2), the definition of η, ξ_i^n, and Remark 3.2, we have

$$\left|y_{\cdot,i}^{\epsilon_n}(r)\right|_{\mathbf{R}^m} \leq \left|y_{\cdot,i}^0\right|_{\mathbf{R}^m} + \int_0^r \begin{bmatrix} E\left|y_{\cdot,i-1}^{\epsilon_n}(s-\epsilon_n)\right|_{\mathbf{R}^m} + 2E\left|y_{\cdot,i}^{\epsilon_n}(s-\epsilon_n)\right|_{\mathbf{R}^m} \\ +E\left|y_{\cdot,i+1}^{\epsilon_n}(s-\epsilon_n)\right|_{\mathbf{R}^m} + c_{1,i} + \eta\left|y_{\cdot,i}^{\epsilon_n}(s-\epsilon_n)\right|_{\mathbf{R}^m} \end{bmatrix} ds$$

$$\leq \left|y_{\cdot,i}^0\right|_{\mathbf{R}^m} + \int_{-\epsilon_n}^{r-\epsilon_n} \begin{bmatrix} E\left|y_{\cdot,i-1}^0(s)\right|_{\mathbf{R}^m} + 2E\left|y_{\cdot,i}^0(s)\right|_{\mathbf{R}^m} \\ +E\left|y_{\cdot,i+1}^0(s)\right|_{\mathbf{R}^m} + c_{1,i} + \eta\left|y_{\cdot,i}^0(s)\right|_{\mathbf{R}^m} \end{bmatrix} ds$$

$$\leq \xi_i^n\left[1+\epsilon_n(1+4E+\eta)\right] := \xi_i^n \mathbf{b}_1 = T_{0,i}^{n,1} = \mathbf{B}_{1,i}^n.$$

Hence,

$$\left|y_{\cdot,i}^{\epsilon_n}(r)\right|_{\mathbf{R}^m} \leq \xi_i^n \mathbf{b}_1, \ \forall r \in [0,\epsilon_n]. \tag{3.20}$$

Although this is not strictly necessary, we shall write the cases $\hat{n} = 2, 3$ in order to illustrate how the induction hypothesis is obtained.

- Case $\hat{n} = 2$.

  Let $r \in]\epsilon_n, 2\epsilon_n]$. Then

$$\left|y_{\cdot,i}^{\epsilon_n}(r)\right|_{\mathbf{R}^m} = \left|y_{\cdot,i}^0 + \int_0^r F_{\cdot,i}\left(y_{\cdot,i}^{\epsilon_n}(s-\epsilon)\right) ds\right|_{\mathbf{R}^m}$$

$$= \left|y_{\cdot,i}^{\epsilon_n}(\epsilon_n) + \int_{\epsilon_n}^r F_{\cdot,i}\left(y_{\cdot,i}^{\epsilon_n}(s-\epsilon)\right) ds\right|_{\mathbf{R}^m}$$

$$\leq \left|y_{\cdot,i}^{\epsilon_n}(\epsilon_n)\right|_{\mathbf{R}^m}$$

$$+ \int_{\epsilon_n}^r \begin{bmatrix} \sum_{j=1}^m \left|a_{\cdot,j} y_{j,i-1}^{\epsilon_n}(s-\epsilon_n)\right|_{\mathbf{R}^m} + 2\sum_{j=1}^m \left|a_{\cdot,j} y_{j,i}^{\epsilon_n}(s-\epsilon_n)\right|_{\mathbf{R}^m} \\ +\sum_{j=1}^m \left|a_{\cdot,j} y_{j,i+1}^{\epsilon_n}(s-\epsilon_n)\right|_{\mathbf{R}^m} + \left|f_{\cdot,i}\left[y_{\cdot,i}^{\epsilon_n}(s-\epsilon_n)\right]\right|_{\mathbf{R}^m} \end{bmatrix} ds. \tag{3.21}$$

  Applying again condition (H2) in (3.21) and arguing as in the previous case, we get

$$\left|y_{\cdot,i}^{\epsilon_n}(r)\right|_{\mathbf{R}^m} \leq \left|y_{\cdot,i}^{\epsilon_n}(\epsilon_n)\right|_{\mathbf{R}^m} + \int_0^{r-\epsilon_n} \begin{bmatrix} E\left|y_{\cdot,i-1}^{\epsilon_n}(s)\right|_{\mathbf{R}^m} + 2E\left|y_{\cdot,i}^{\epsilon_n}(s)\right|_{\mathbf{R}^m} \\ +E\left|y_{\cdot,i+1}^{\epsilon_n}(s)\right|_{\mathbf{R}^m} + c_{1,i} + \eta\left|y_{\cdot,i}^{\epsilon_n}(s)\right|_{\mathbf{R}^m} \end{bmatrix} ds.$$

  Now, (3.20) yields

$$\left|y_{\cdot,i}^{\epsilon_n}(r)\right|_{\mathbf{R}^m} \leq \xi_i^n \mathbf{b}_1 + \epsilon_n\left[E\xi_{i-1}^n \mathbf{b}_1 + 2E\xi_i^n \mathbf{b}_1 + E\xi_{i+1}^n \mathbf{b}_1 + \xi_i^n + \eta\xi_i^n \mathbf{b}_1\right]$$

$$\leq \xi_i^n \mathbf{b}_1\left[1+\epsilon_n(2E+\eta)\right] + \epsilon_n\xi_i^n + \epsilon_n E\left[\xi_{i-1}^n \mathbf{b}_1 + \xi_{i+1}^n \mathbf{b}_1\right]$$

$$\leq \xi_i^n\left[1+(1+4E+\eta)\epsilon_n\right]\left[1+\epsilon_n(2E+\eta)\right] + \epsilon_n\xi_i^n +$$

$$+ \epsilon_n E\left[\xi_{i-1}^n \mathbf{b}_1 + \xi_{i+1}^n \mathbf{b}_1\right]$$

$$\leq \xi_i^n \left[1 + \epsilon_n (2 + 6E + 2\eta) + \epsilon_n^2 (2E + \eta)(1 + 4E + \eta)\right] + \\ + \epsilon_n E \left[\xi_{i-1}^n \mathbf{b}_1 + \xi_{i+1}^n \mathbf{b}_1\right].$$

Thus,

$$\begin{aligned}\left|y_{\cdot,i}^{\epsilon_n}(r)\right|_{\mathbf{R}^m} &\leq \xi_i^n \left[1 + \epsilon_n (2 + 6E + 2\eta) + \epsilon_n^2 (2E + \eta)(1 + 4E + \eta)\right] \\ &\quad + \epsilon_n E \left\{\xi_{i-1}^n + \xi_{i+1}^n\right\} [1 + \epsilon_n (1 + 4E + \eta)] \\ &= T_{0,i}^{n,2} + T_{1,i}^{n,2} := B_{2,i}^n.\end{aligned}$$

- Case $\hat{n} = 3$.

  Let $r \in]2\epsilon_n, 3\epsilon_n]$. Repeating similar arguments as before, we have

$$\begin{aligned}\left|y_{\cdot,i}^{\epsilon_n}(r)\right|_{\mathbf{R}^m} &\leq \mathbf{B}_{2,i}^n + \epsilon_n E \mathbf{B}_{2,i-1}^n + 2\epsilon_n E \mathbf{B}_{2,i}^n + \epsilon_n E \mathbf{B}_{2,i+1}^n + \epsilon_n \eta \mathbf{B}_{2,i}^n + \epsilon_n \xi_i^n \\ &\leq \mathbf{B}_{2,i}^n [1 + \epsilon_n (2E + \eta)] + \epsilon_n \xi_i^n + \epsilon_n E \left[\mathbf{B}_{2,i-1}^n + \mathbf{B}_{2,i+1}^n\right] \\ &\leq \xi_i^n [1 + \epsilon_n (2E + \eta)] [1 + \epsilon_n (2 + 6E + 2\eta) \\ &\quad + \epsilon_n^2 (2E + \eta)(1 + 4E + \eta)] \\ &\quad + \epsilon_n \xi_i^n + \epsilon_n E [1 + \epsilon_n (2E + \eta)] \left[\mathbf{B}_{1,i-1}^n + \mathbf{B}_{1,i+1}^n\right] \\ &\quad + \epsilon_n E \left[\mathbf{B}_{2,i-1}^n + \mathbf{B}_{2,i+1}^n\right].\end{aligned} \tag{3.22}$$

Let us analyze separately each of the terms in (3.22). First,

$$\begin{aligned}&\xi_i^n [1 + \epsilon_n (2E + \eta)] \left[1 + \epsilon_n (2 + 6E + 2\eta) + \epsilon_n^2 (2E + \eta)(1 + 4E + \eta)\right] + \epsilon_n \xi_i^n \\ &\quad = \xi_i^n \left\{\begin{array}{c} 1 + \epsilon_n (1 + 2E + \eta) + \left(\epsilon_n + \epsilon_n^2 (2E + \eta)\right)(2 + 6E + 2\eta) \\ + (2E + \eta)(1 + 4E + \eta)\left(\epsilon_n^2 + \epsilon_n^3 (2E + \eta)\right)\end{array}\right\} \\ &\quad = \xi_i^n \left\{\begin{array}{c} 1 + \epsilon_n (3 + 8E + 3\eta) + \epsilon_n^2 (2E + \eta)(3 + 10E + 3\eta) \\ + \epsilon_n^3 (2E + \eta)^2 (1 + 4E + \eta)\end{array}\right\}.\end{aligned} \tag{3.23}$$

For the second term, we have

$$\begin{aligned}&\epsilon_n E [1 + \epsilon_n (2E + \eta)] \left[\mathbf{B}_{1,i-1}^n + \mathbf{B}_{1,i+1}^n\right] + \epsilon_n E \left[\mathbf{B}_{2,i-1}^n + \mathbf{B}_{2,i+1}^n\right] = \\ &\quad = \epsilon_n E [1 + \epsilon_n (2E + \eta)] [1 + \epsilon_n (1 + 4E + \eta)] \left[\xi_{i-1}^n + \xi_{i+1}^n\right] \\ &\qquad + \epsilon_n E \left[\mathbf{B}_{2,i-1}^n + +\mathbf{B}_{2,i+1}^n\right] \\ &\quad = \epsilon_n E \left[1 + \epsilon_n (1 + 6E + 2\eta) + \epsilon_n^2 (2E + \eta)(1 + 4E + \eta)\right] \left[\xi_{i-1}^n + \xi_{i+1}^n\right] \\ &\qquad + \epsilon_n E \left[\mathbf{B}_{2,i-1}^n + \mathbf{B}_{2,i+1}^n\right].\end{aligned}$$

Finally, we estimate $\epsilon_n E\left[\mathbf{B}^n_{2,i-1} + \mathbf{B}^n_{2,i+1}\right]$. By definition,

$$\begin{aligned}\mathbf{B}^n_{2,i-1} &:= \xi^n_{i-1}\left[1 + \epsilon_n (2 + 6E + 2\eta) + \epsilon_n^2 (2E + \eta)(1 + 4E + \eta)\right] \\ &\quad + \epsilon_n E\left[1 + (1 + 4E + \eta)\epsilon_n\right]\left[\xi^n_{i-2} + \xi^n_i\right], \\ \mathbf{B}^n_{2,i+1} &:= \xi^n_{i+1}\left[1 + \epsilon_n (2 + 6E + 2\eta) + \epsilon_n^2 (2E + \eta)(1 + 4E + \eta)\right] \\ &\quad + \epsilon_n E\left[1 + (1 + 4E + \eta)\epsilon_n\right]\left[\xi^n_i + \xi^n_{i+2}\right],\end{aligned}$$

so

$$\begin{aligned}\epsilon_n E\left[\mathbf{B}_{2,i-1} + \mathbf{B}_{2,i+1}\right] &\le \epsilon_n E\left[\xi^n_{i-1} + \xi^n_{i+1}\right]\begin{bmatrix} 1 + \epsilon_n (2 + 6E + 2\eta) \\ +\epsilon_n^2 (2E + \eta)(1 + 4E + \eta)\end{bmatrix} \\ &\quad + \epsilon_n^2 E^2\left[\xi^n_{i-2} + 2\xi^n_i + \xi^n_{i+2}\right]\left[1 + (1 + 4E + \eta)\epsilon_n\right].\end{aligned} \tag{3.24}$$

Combining (3.22)–(3.24), we have

$$\begin{aligned}\left|y^{\epsilon_n}_{\cdot,i}(r)\right|_{\mathbf{R}^m} &\le \xi^n_i\left\{\begin{matrix} 1 + \epsilon_n (3 + 8E + 3\eta) + \epsilon_n^2 (2E + \eta)(3 + 10E + 3\eta) \\ + \epsilon_n^3 (2E + \eta)^2 (1 + 4E + \eta)\end{matrix}\right\} \\ &\quad + \epsilon_n E\left[\xi^n_{i-1} + \xi^n_{i+1}\right] \\ &\quad \times\begin{bmatrix} 1 + \epsilon_n (1 + 6E + 2\eta) + \epsilon_n^2 (2E + \eta)(1 + 4E + \eta) + \\ 1 + \epsilon_n (2 + 6E + 2\eta) + \epsilon_n^2 (2E + \eta)(1 + 4E + \eta)\end{bmatrix} \\ &\quad + \epsilon_n^2 E^2\left\{\xi^n_{i-2} + 2\xi^n_i + \xi^n_{i+2}\right\}\left[1 + (1 + 4E + \eta)\epsilon_n\right] \\ &\le \xi^n_i\left\{\begin{matrix} 1 + \epsilon_n (3 + 8E + 3\eta) + \epsilon_n^2 (2E + \eta)(3 + 10E + 3\eta) \\ +\epsilon_n^3 (2E + \eta)^2 (1 + 4E + \eta)\end{matrix}\right\} \\ &\quad +2\epsilon_n E\left[\xi^n_{i-1} + \xi^n_{i+1}\right]\left[1 + \epsilon_n (2 + 6E + 2\eta)\right. \\ &\quad \left.+\epsilon_n^2 (2E + \eta)(1 + 4E + \eta)\right] \\ &\quad + \epsilon_n^2 E^2\left\{\xi^n_{i-2} + 2\xi^n_i + \xi^n_{i+2}\right\}\left[1 + (1 + 4E + \eta)\epsilon_n\right] = \mathbf{B}^n_{3,i}.\end{aligned}$$

Consider now the general case.

- Case $\hat{\mathbf{n}} + \mathbf{1}$.

  Let $r \in \left]\hat{n}\epsilon_n, (\hat{n} + 1)\epsilon_n\right]$ and assume that estimate (3.2) is true for $r \in \left](\hat{n} - 1)\epsilon_n, \hat{n}\epsilon_n\right]$.

  We need to check that $\left|y^{\epsilon_n}_{\cdot,i}(r)\right|_{\mathbf{R}^m} \le \sum_{p=0}^{\hat{n}} T^{n,\hat{n}+1}_{p,i} = \mathbf{B}^n_{\hat{n}+1,i}$. First, by (H2) and Remark 3.2, we get

$$\begin{aligned}\left|y_{\cdot,i}^{\epsilon_n}(r)\right|_{\mathbf{R}^m} &\le \left|y_{\cdot,i}^{\epsilon_n}(\hat{n}\epsilon_n)\right|_{\mathbf{R}^m} + \int_{\hat{n}\epsilon_n}^{r}\left|F_{\cdot,i}\left(y_{\cdot,i}^{\epsilon_n}(s-\epsilon_n)\right)\right|_{\mathbf{R}^m} ds\\ &\le \left|y_{\cdot,i}^{\epsilon_n}(\hat{n}\epsilon_n)\right|_{\mathbf{R}^m} + \int_{\hat{n}\epsilon_n}^{r}\left\{c_{1,i} + \eta\left|y_{\cdot,i}^{\epsilon_n}(s-\epsilon_n)\right|_{\mathbf{R}^m}\right\} ds\\ &\quad + E\int_{\hat{n}\epsilon_n}^{r}\left\{\left|y_{\cdot,i-1}^{\epsilon_n}(s-\epsilon_n)\right|_{\mathbf{R}^m} + 2\left|y_{\cdot,i}^{\epsilon_n}(s-\epsilon_n)\right|_{\mathbf{R}^m}\right.\\ &\quad \left. + \left|y_{\cdot,i+1}^{\epsilon_n}(s-\epsilon_n)\right|_{\mathbf{R}^m}\right\} ds.\end{aligned}$$

Since $(s-\epsilon_n) \in](\hat{n}-1)\epsilon_n, \hat{n}\epsilon_n]$, we can use the induction hypothesis. Hence,

$$\begin{aligned}\left|y_{\cdot,i}^{\epsilon_n}(r)\right|_{\mathbf{R}^m} &\le \mathbf{B}_{\hat{n},i}^{n} + (\epsilon_n E)\left\{\mathbf{B}_{\hat{n},i-1}^{n} + 2\mathbf{B}_{\hat{n},i}^{n} + \mathbf{B}_{\hat{n},i+1}^{n}\right\} + \epsilon_n\left\{\xi_i^n + \eta\mathbf{B}_{\hat{n},i}^{n}\right\}\\ &\le \mathbf{B}_{\hat{n},i}^{n}\left\{1 + \epsilon_n(2E+\eta)\right\} + \epsilon_n\xi_i^n + (\epsilon_n E)\left\{\mathbf{B}_{\hat{n},i-1}^{n} + \mathbf{B}_{\hat{n},i+1}^{n}\right\},\end{aligned}$$

and using the definition of $T_{p,i}^{n,\hat{n}}$, $T_{p,i-1}^{n,\hat{n}}$ and $T_{p,i+1}^{n,\hat{n}}$, we have

$$\begin{aligned}\left|y_{\cdot,i}^{\epsilon_n}(r)\right|_{\mathbf{R}^m} &\le \sum_{p=0}^{\hat{n}-1} T_{p,i}^{n,\hat{n}} + \epsilon_n\xi_i^n + \epsilon_n(2E+\eta)\sum_{p=0}^{\hat{n}-1} T_{p,i}^{n,\hat{n}}\\ &\quad + (\epsilon_n E)\sum_{p=0}^{\hat{n}-1}\left\{T_{p,i-1}^{n,\hat{n}} + T_{p,i+1}^{n,\hat{n}}\right\}.\end{aligned}$$

To complete the proof, we need to obtain the inequality

$$\begin{aligned}&\sum_{p=0}^{\hat{n}-1} T_{p,i}^{n,\hat{n}} + \epsilon_n\xi_i^n + \epsilon_n(2E+\eta)\sum_{p=0}^{\hat{n}-1} T_{p,i}^{n,\hat{n}} + (\epsilon_n E)\sum_{p=0}^{\hat{n}-1}\left\{T_{p,i-1}^{n,\hat{n}} + T_{p,i+1}^{n,\hat{n}}\right\}\\ &\quad \le \sum_{p=0}^{\hat{n}} T_{p,i}^{n,\hat{n}+1}.\end{aligned}$$

This follows from the following:

$$T_{0,i}^{n,\hat{n}+1} = T_{0,i}^{n,\hat{n}} + \epsilon_n(2E+\eta)\,T_{0,i}^{n,\hat{n}} + \epsilon_n\xi_i^n, \tag{3.25}$$

$$T_{k,i}^{n,\hat{n}+1} \ge T_{k,i}^{n,\hat{n}} + \epsilon_n(2E+\eta)\,T_{k,i}^{n,\hat{n}} + \epsilon_n E\left\{T_{k-1,i-1}^{n,\hat{n}} + T_{k-1,i+1}^{n,\hat{n}}\right\}, \tag{3.26}$$

for $k = 1, 2, \ldots, \hat{n}-1$,

$$T_{\hat{n},i}^{n,\hat{n}+1} = \epsilon_n E \left\{ T_{\hat{n}-1,i-1}^{n,\hat{n}} + T_{\hat{n}-1,i+1}^{n,\hat{n}} \right\}. \tag{3.27}$$

Let us consider (3.25). The expression $T_{0,i}^{n,\hat{n}} + \epsilon_n (2E + \eta) T_{0,i}^{n,\hat{n}} + \epsilon_n \xi_i^n$ can be rewritten as follows:

$$T_{0,i}^{n,\hat{n}} + \epsilon_n (2E + \eta) T_{0,i}^{n,\hat{n}} + \epsilon_n \xi_i^n = (\epsilon_n E)^0 \binom{\hat{n}-1}{0} \xi_i^n$$

$$\cdot \left[\begin{array}{c} \binom{\hat{n}}{0} \\ +\epsilon_n \left\{ \binom{\hat{n}}{0} + \binom{\hat{n}}{1} + \left\{ \begin{bmatrix} \hat{n} \\ 0 \end{bmatrix} + \begin{bmatrix} \hat{n} \\ 1 \end{bmatrix} \right\} E \right. \\ \left. + \left\{ \binom{\hat{n}}{0} + \binom{\hat{n}}{1} \right\} \eta \right\} \\ +\epsilon_n^2 \left\{ \binom{\hat{n}}{1} + \binom{\hat{n}}{2} + \left\{ \begin{bmatrix} \hat{n} \\ 1 \end{bmatrix} + \begin{bmatrix} \hat{n} \\ 2 \end{bmatrix} \right\} E \right. \\ \left. + \left\{ \binom{\hat{n}}{1} + \binom{\hat{n}}{2} \right\} \eta \right\} (2E + \eta) \\ +\epsilon_n^3 \left\{ \binom{\hat{n}}{2} + \binom{\hat{n}}{3} + \left\{ \begin{bmatrix} \hat{n} \\ 2 \end{bmatrix} + \begin{bmatrix} \hat{n} \\ 3 \end{bmatrix} \right\} E \right. \\ \left. + \left\{ \binom{\hat{n}}{2} + \binom{\hat{n}}{3} \right\} \eta \right\} (2E + \eta)^2 \\ + \dots \\ +\epsilon_n^{\hat{n}} \left\{ \binom{\hat{n}}{\hat{n}-1} + \binom{\hat{n}}{\hat{n}} + \left\{ \begin{bmatrix} \hat{n} \\ \hat{n}-1 \end{bmatrix} + \begin{bmatrix} \hat{n} \\ \hat{n} \end{bmatrix} \right\} E \right. \\ \left. + \left\{ \binom{\hat{n}}{\hat{n}-1} + \binom{\hat{n}}{\hat{n}} \right\} \eta \right\} (2E + \eta)^{\hat{n}-1} \\ +\epsilon_n^{\hat{n}+1} \left\{ \binom{\hat{n}}{\hat{n}} + \begin{bmatrix} \hat{n} \\ \hat{n} \end{bmatrix} E + \binom{\hat{n}}{\hat{n}} \eta \right\} (2E + \eta)^{\hat{n}} \end{array} \right], \tag{3.28}$$

where we have used that $\begin{bmatrix} \hat{n} \\ 0 \end{bmatrix} = 2 \binom{\hat{n}}{0}$. Using $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$, $\binom{n}{n} = \binom{n+1}{n+1}$ and $\binom{n}{0} = \binom{n+1}{0}$, jointly with (3.10), we obtain that

$$
T_{0,i}^{n,\hat{n}+1} = (\epsilon_n E)^0 \binom{\hat{n}-1}{0} \xi_i^n
$$

$$
\cdot \left[\begin{array}{c} \binom{\hat{n}+1}{0} \\ +\epsilon_n \left\{ \binom{\hat{n}+1}{1} + \begin{bmatrix} \hat{n}+1 \\ 1 \end{bmatrix} E + \binom{\hat{n}+1}{1} \eta \right\} \\ +\epsilon_n^2 \left\{ \binom{\hat{n}+1}{2} + \begin{bmatrix} \hat{n}+1 \\ 2 \end{bmatrix} E + \binom{\hat{n}+1}{2} \eta \right\} (2E+\eta) \\ \epsilon_n^3 \left\{ \binom{\hat{n}+1}{3} + \begin{bmatrix} \hat{n}+1 \\ 3 \end{bmatrix} E + \binom{\hat{n}+1}{3} \eta \right\} (2E+\eta)^2 \\ + \dots \\ +\epsilon_n^{\hat{n}+1} \left\{ \binom{\hat{n}+1}{\hat{n}+1} + \begin{bmatrix} \hat{n}+1 \\ \hat{n}+1 \end{bmatrix} E + \binom{\hat{n}+1}{\hat{n}+1} \eta \right\} (2E+\eta)^{\hat{n}} \end{array} \right], \tag{3.29}
$$

is the same as in (3.28). Thus, (3.25) is proved.

Further, let us prove (3.26). Arguing as before, we can obtain that

$$
\begin{aligned}
&T_{k,i}^{n,\hat{n}} + \epsilon_n (2E+\eta) T_{k,i}^{n,\hat{n}} \\
&\quad \le (\epsilon_n E)^k \left\{ \binom{k}{0} \xi_{i-k}^n + \binom{k}{1} \xi_{i-k+2}^n + \dots + \binom{k}{k} \xi_{i+k}^n \right\} \binom{\hat{n}-1}{k}
\end{aligned}
$$

$$
\cdot \left[\begin{array}{c} \binom{\hat{n}+1-k}{0} \\ +\epsilon_n \left\{ \binom{\hat{n}+1-k}{1} + \begin{bmatrix} \hat{n}+1-k \\ 1 \end{bmatrix} E + \binom{\hat{n}+1-k}{1} \eta \right\} \\ +\epsilon_n^2 \left\{ \binom{\hat{n}+1-k}{2} + \begin{bmatrix} \hat{n}+1-k \\ 2 \end{bmatrix} E + \binom{\hat{n}+1-k}{2} \eta \right\} (2E+\eta) \\ + \dots \\ +\epsilon_n^{\hat{n}+1-k} \left\{ \binom{\hat{n}+1-k}{\hat{n}+1-k} + \begin{bmatrix} \hat{n}+1-k \\ \hat{n}+1-k \end{bmatrix} E + \binom{\hat{n}+1-k}{\hat{n}+1-k} \eta \right\} (2E+\eta)^{\hat{n}-k} \end{array} \right],
$$

$$
\begin{aligned}
&\epsilon_n E \left\{ T_{k-1,i-1}^{n,\hat{n}} + T_{k-1,i+1}^{n,\hat{n}} \right\} \\
&\quad = (\epsilon_n E)^k \left\{ \binom{k}{0} \xi_{i-k}^n + \binom{k}{1} \xi_{i-k+2}^n + \dots + \binom{k}{k} \xi_{i+k}^n \right\} \binom{\hat{n}-1}{k-1}
\end{aligned}
$$

$$\cdot\left[\begin{array}{c}\binom{\hat{n}+1-k}{0}\\ +\epsilon_n\left\{\binom{\hat{n}+1-k}{1}+\left[\begin{matrix}\hat{n}+1-k\\ 1\end{matrix}\right]E+\binom{\hat{n}+1-k}{1}\eta\right\}\\ +\epsilon_n^2\left\{\binom{\hat{n}+1-k}{2}+\left[\begin{matrix}\hat{n}+1-k\\ 2\end{matrix}\right]E+\binom{\hat{n}+1-k}{2}\eta\right\}(2E+\eta)\\ +\ldots\\ +\epsilon_n^{\hat{n}+1-k}\left\{\binom{\hat{n}+1-k}{\hat{n}+1-k}+\left[\begin{matrix}\hat{n}+1-k\\ \hat{n}+1-k\end{matrix}\right]E+\binom{\hat{n}+1-k}{\hat{n}+1-k}\eta\right\}(2E+\eta)^{\hat{n}-k}\end{array}\right].$$

Summing these expressions, using $\binom{\hat{n}-1}{k-1}+\binom{\hat{n}-1}{k}=\binom{\hat{n}}{k}$ and developing $T_{k,i}^{n,\hat{n}+1}$ in a similar way as in (3.29), we obtain inequality (3.26).

The proof of (3.27) is easier than the preceding ones. We use here similar arguments but now over the term $\epsilon_n E\{T_{\hat{n}-1,i-1}^{n,\hat{n}}+T_{\hat{n}-1,i+1}^{n,\hat{n}}\}$.

Now, the proof of Lemma 3.2 is finished. □

We return to the arguments previous to Lemma 3.2.

Let $\xi>0$ be arbitrary. As $w\in l_m^2$, we can find $K_1(\xi)\in\mathbf{Z}$ such that $\sum_{j=1}^m\sum_{|i|>K_1}\left|w_{j,i}\right|^2\leq\frac{\xi}{8}$. Take $K_2\geq K_1$. Then

$$\begin{aligned}\left\|y^{\epsilon_n}(t^*)-w\right\|_{l_m^2}^2&=\sum_{j=1}^m\sum_{|i|\leq K_2}\left|y_{j,i}^{\epsilon_n}(t^*)-w_{j,i}\right|^2+\sum_{j=1}^m\sum_{|i|>K_2}\left|y_{j,i}^{\epsilon_n}(t^*)-w_{j,i}\right|^2\\&\leq\sum_{j=1}^m\sum_{|i|\leq K_2}\left|y_{j,i}^{\epsilon_n}(t^*)-w_{j,i}\right|^2\\&\quad+2\sum_{j=1}^m\left(\sum_{|i|>K_2}\left|y_{j,i}^{\epsilon_n}(t^*)\right|^2+\sum_{|i|>K_2}\left|w_{j,i}\right|^2\right)\\&\leq\frac{\xi}{4}+2\sum_{j=1}^m\sum_{|i|>K_2}\left|y_{j,i}^{\epsilon_n}(t^*)\right|^2+\sum_{j=1}^m\sum_{|i|\leq K_2}\left|y_{j,i}^{\epsilon_n}(t^*)-w_{j,i}\right|^2.\end{aligned}\tag{3.30}$$

Let us check the existence of $K_2(\xi)$ and $N(\xi)$ such that

$$\sum_{j=1}^m\sum_{|i|>K_2}\left|y_{j,i}^{\epsilon_n}(t^*)\right|^2\leq\frac{\xi}{4},\ \sum_{j=1}^m\sum_{|i|\leq K_2}\left|y_{j,i}^{\epsilon_n}(t^*)-w_{j,i}\right|^2\leq\frac{\xi}{4},\ \forall n\geq N.$$

First, we shall estimate the second term in (3.30). After that, for the third one, we use the fact that the weakly convergence in l_m^2 implies strong convergence on any $\mathbf{R}_m^d := \{(x_1 \ \dots \ x_m) : x_i \in \mathbf{R}^d, \ \forall i\}$, where $d < \infty$. For this aim, we will estimate each term in (3.19).

Let $\hat{n}$ be such that $t^* \in](\hat{n}-1)\,\epsilon_n, \hat{n}\epsilon_n]$. Using (3.19) and property (3.11), we have

$$\left|y_{\cdot,i}^{\epsilon_n}(r)\right|_{\mathbf{R}^m} \leq \sum_{p=0}^{\hat{n}-1} (\epsilon_n E)^p \binom{\hat{n}-1}{p}$$

$$\times \left\{ \binom{p}{0} \xi_{i-p} + \binom{p}{1} \xi_{i-p+2} + \dots + \binom{p}{p} \xi_{i+p} \right\}$$

$$\cdot \left[\begin{array}{c} \binom{\hat{n}-p}{0} + \epsilon_n \binom{\hat{n}-p}{1} \{1+4E+\eta\} + \epsilon_n^2 \binom{\hat{n}-p}{2} \{1+4E+\eta\}(2E+\eta) \\ + \dots + \epsilon_n^{\hat{n}-p} \binom{\hat{n}-p}{\hat{n}-p} \{1+4E+\eta\}(2E+\eta)^{\hat{n}-p-1} \end{array} \right],$$

for all $r \in](\hat{n}-1)\,\epsilon_n, \hat{n}\epsilon_n]$.

By $\hat{n}\epsilon_n \leq \hat{\alpha}$ and (3.12), we obtain $\binom{\hat{n}}{k} \epsilon_n^k \leq \hat{n}^k \epsilon_n^k \leq \hat{\alpha}^k$. Thus,

$$\left|y_{\cdot,i}^{\epsilon_n}(r)\right|_{\mathbf{R}^m} \leq \sum_{p=0}^{\hat{n}-1} (\epsilon_n E)^p \binom{\hat{n}-1}{p}$$

$$\times \left\{ \binom{p}{0} \xi_{i-p}^n + \binom{p}{1} \xi_{i-p+2}^n + \dots + \binom{p}{p} \xi_{i+p}^n \right\}$$

$$\cdot \left[\begin{array}{c} 1 + (\hat{n}-p)\,\epsilon_n \{1+4E+\eta\} + [(\hat{n}-p)\,\epsilon_n]^2 \{1+4E+\eta\}(2E+\eta) \\ + \dots + [(\hat{n}-p)\,\epsilon_n]^{\hat{n}-p} \{1+4E+\eta\}(2E+\eta)^{\hat{n}-p-1} \end{array} \right]$$

$$\leq \sum_{p=0}^{\hat{n}-1} \left[(\epsilon_n E)^p \binom{\hat{n}-1}{p} \left\{ \binom{p}{0} \xi_{i-p}^n + \binom{p}{1} \xi_{i-p+2}^n + \dots + \binom{p}{p} \xi_{i+p}^n \right\} \right.$$

$$\left. \times \sum_{j=0}^{\hat{n}-p} \hat{\alpha}^j \,(1+4E+\eta)^j \right] \tag{3.31}$$

Using $\beta = 2\hat{\alpha}\,(1+4E+\eta) < 1$, we obtain that $\sum_{j=0}^{\hat{n}-p} \hat{\alpha}^j \,(1+4E+\eta)^j \leq \frac{1}{1-\beta}$. Defining the sequence $\eta_{i,p} := \max\left\{\xi_{i-p}^n, \xi_{i-p+2}^n, \dots, \xi_{i+p}^n\right\}$, with $p = 1, \dots, \hat{n}-1$, and using $\binom{\hat{n}-1}{p} \leq \hat{n}^p$, we can estimate (3.31) as follows:

$$\begin{aligned}\left|y_{\cdot,i}^{\epsilon_n}(r)\right|_{\mathbf{R}^m} &\leq \tfrac{1}{1-\beta}\left[\xi_i^n + \textstyle\sum_{p=1}^{\hat{n}-1}(\epsilon_n E)^p \binom{\hat{n}-1}{p}\eta_{i,p}2^p\right] \\ &\leq \tfrac{1}{1-\beta}\left(\xi_i^n + \textstyle\sum_{p=1}^{\hat{n}-1}(2\hat{n}\epsilon_n E)^p\,\eta_{i,p}\right) \leq \tfrac{1}{1-\beta}\left(\xi_i^n + \textstyle\sum_{p=1}^{\hat{n}-1}\beta^p\eta_{i,p}\right),\end{aligned}$$

as $2\hat{n}\epsilon_n E \leq \beta$.

Let us take some $\hat{K} \leq \hat{n} - 1$ (which will be determined later on). Then,

$$\begin{aligned}\left|y_{\cdot,i}^{\epsilon_n}(r)\right|_{\mathbf{R}^m} &\leq \tfrac{1}{1-\beta}\left(\xi_i^n + \textstyle\sum_{p=1}^{\hat{K}-2}\beta^p\eta_{i,p} + \sum_{p=\hat{K}-1}^{\hat{n}-1}\beta^p\eta_{i,p}\right) \\ &= \tfrac{1}{1-\beta}\left(\xi_i^n + \textstyle\sum_{p=1}^{\hat{K}-2}\beta^p\eta_{i,p} + \sum_{p=0}^{\hat{n}-\hat{K}}\beta^{p+\hat{K}-1}\eta_{i,p+\hat{K}-1}\right),\end{aligned}$$

so that

$$\begin{aligned}\sum_{|i|\geq G}\left|y_{\cdot,i}^{\epsilon_n}(r)\right|_{\mathbf{R}^m}^2 \leq \frac{3}{(1-\beta)^2}&\left(\sum_{|i|\geq G}\left(\xi_i^n\right)^2 + \sum_{|i|\geq G}\left[\sum_{p=1}^{\hat{K}-2}\beta^p\eta_{i,p}\right]^2\right. \\ &\left.+\sum_{|i|\geq G}\left[\sum_{p=0}^{\hat{n}-\hat{K}}\beta^{p+\hat{K}-1}\eta_{i,p+\hat{K}-1}\right]^2\right).\end{aligned}$$

Since $\beta < 1$ and $\left(a_1 + \ldots + a_p\right)^2 \leq p\left(a_1^2 + \ldots + a_p^2\right)$, we have

$$\begin{aligned}\sum_{|i|}\geq G\left|y_{\cdot,i}^{\epsilon_n}(r)\right|_{\mathbf{R}^m}^2 \leq \frac{3}{(1-\beta)^2}&\left(\sum_{|i|\geq G}\left(\xi_i^n\right)^2\right. \\ &\left.+\hat{K}\sum_{|i|\geq G}\sum_{p=1}^{\hat{K}-2}\eta_{i,p}^2 + \beta^{2(\hat{K}-1)}\sum_{|i|\geq G}\left[\sum_{p=0}^{\hat{n}-\hat{K}}\beta^p\eta_{i,p+\hat{K}-1}\right]^2\right).\end{aligned} \tag{3.32}$$

We shall estimate each of the three terms in the last inequality. First, we note that

$$\begin{aligned}\textstyle\sum_{|i|\geq G}\eta_{i,p}^2 &\leq \textstyle\sum_{|i|\geq G}\left\{\left(\xi_{i-p}^n\right)^2 + \left(\xi_{i-p+1}^n\right)^2 + \overset{(2p+1)}{\cdots} + \left(\xi_{i+p}^n\right)^2\right\} \\ &\leq (2p+1)\textstyle\sum_{|i|\geq G-p}\left(\xi_i^n\right)^2 \\ &\leq (2p+1)\textstyle\sum_{i\in\mathbf{Z}}\left(\xi_i^n\right)^2 \leq (2p+1)\,C,\end{aligned} \tag{3.33}$$

where we have used that

$$C_{G,p} = \sum_{|i|\geq G-p}\left(\xi_i^n\right)^2 \leq \sum_{i\in\mathbf{Z}}\left(\xi_i^n\right)^2 := C\left(\{\xi_i^n\}\right) \leq \sum_{i\in\mathbf{Z}}\left(\xi_i^0\right)^2 := C.$$

Then

$$\begin{aligned}
&\sum_{|i|\geq G}\left[\sum_{p=0}^{\hat{n}-\hat{K}}\beta^{p}\eta_{i,p+\hat{K}-1}\right]^{2}\\
&=\sum_{|i|\geq G}\sum_{s=0}^{\hat{n}-\hat{K}}\sum_{p=0}^{\hat{n}-\hat{K}}\beta^{p+s}\eta_{i,p+\hat{K}-1}\eta_{i,s+\hat{K}-1}\\
&\leq\sum_{s=0}^{\hat{n}-\hat{K}}\sum_{p=0}^{\hat{n}-\hat{K}}\sum_{|i|\geq G}\frac{\beta^{p+s}}{2}\left(\eta^{2}_{i,p+\hat{K}-1}+\eta^{2}_{i,s+\hat{K}-1}\right)\\
&\leq\frac{C}{2}\sum_{p=0}^{\hat{n}-\hat{K}}\sum_{s=0}^{\hat{n}-\hat{K}}\beta^{s+p}\left[\left(2p+2\hat{K}-1\right)+\left(2s+2\hat{K}-1\right)\right].
\end{aligned}$$

Putting $a_p = 2\hat{K} + 2p - 1$, we obtain

$$\begin{aligned}
\sum_{|i|\geq G}\left[\sum_{p=0}^{\hat{n}-\hat{K}}\beta^{p}\eta_{i,p+\hat{K}-1}\right]^{2}&\leq\frac{C}{2}\sum_{p=0}^{\hat{n}-\hat{K}}\sum_{s=0}^{\hat{n}-\hat{K}}\beta^{s+p}\left(a_s+a_p\right)\\
&=C\sum_{p=0}^{\hat{n}-\hat{K}}\beta^{p}a_p\sum_{s=0}^{\hat{n}-\hat{K}}\beta^{s}\\
&\leq\frac{C}{1-\beta}\left[\frac{2\hat{K}-1}{1-\beta}+\frac{2\beta}{(1-\beta)^{2}}\right].
\end{aligned}$$

Hence, we can choose $\hat{K}$ big enough, so that

$$\frac{3}{(1-\beta)^{2}}\beta^{2(\hat{K}-1)}\sum_{|i|\geq G}\left[\sum_{p=0}^{\hat{n}-\hat{K}}\beta^{p}\eta_{i,p+\hat{K}-1}\right]^{2}<\frac{\xi}{12}. \tag{3.34}$$

We note that $\hat{n}(n)\to\infty$ as $n\to\infty$. Then there exists $N_0(\xi)$ such that $\hat{K}\leq\hat{n}-1$, $\forall n\geq N_0$.

Further, we shall analyze the second term in (3.32). Using (3.33), we have

$$\begin{aligned}
\hat{K}\sum_{|i|\geq G}\sum_{p=1}^{\hat{K}-2}\eta^{2}_{i,p}&\leq\hat{K}\sum_{|i|\geq G}\sum_{p=1}^{\hat{K}}\eta^{2}_{i,\hat{K}}\leq\hat{K}^{2}\sum_{|i|\geq G}\eta^{2}_{i,\hat{K}}\\
&\leq\hat{K}^{2}\left(2\hat{K}+1\right)\sum_{|i|\geq G-\hat{K}}\left(\xi_i^{n}\right)^{2}=\hat{K}^{2}\left(2\hat{K}+1\right)C_{G,\hat{K}}.
\end{aligned} \tag{3.35}$$

We need the following lemma:

Lemma 3.3. *For any* $0<\tau<1$, *we can find* $\delta(\tau)$, $K_0(\tau)$, $N_1(\tau)$ *such that*

$$\sum_{j=1}^{m}\sum_{|i|>K_0}\left|y^{0}_{j,i}(t)\right|^{2}\leq\tau,\ \forall\, t\in[-\delta(\tau),0]\ \textit{and} \tag{3.36}$$

$$\sum_{j=1}^{m}\sum_{|i|\geq K_0}\left|y^{0}_{j,i}(t)\right|^{2}\leq\tau,\ \forall t\in[-\epsilon_n,0],\ n\geq N_1. \tag{3.37}$$

Proof. Since y^0 is continuous, there exists $\delta(\tau)$ such that

$$\left\|y^{0}(t)-y^{0}(0)\right\|_{l^2_m}=\left\|y^{0}(t)-y^{0}\right\|_{l^2_m}\leq\frac{\tau}{4},\ \forall t\in[-\delta(\tau),0].$$

Moreover, $y^0 \in l_m^2$ implies that we can find K_0 for which $\sum_{j=1}^{m} \sum_{|i|>K_0} \left| y_{j,i}^0 \right|^2 \leq \frac{\tau}{4}$. Thus, (3.36) follows.

In a similar way, and using that for a fixed $\delta(\tau)$, there exists $N_1(\tau)$ such that $\epsilon_n < \delta(\tau)$ if $n \geq N_1(\tau)$, (3.37) follows. □

Now, by Lemma 3.3 and the definition of ξ_i^n, one can find $G_0(\xi)$, $G_1(\xi)$, and $N_1(\xi) \geq N_0$ such that

$$\frac{3}{(1-\beta)^2} \hat{K}^2 \left(2\hat{K}+1\right) \sum_{|i| \geq G-\hat{K}} \left(\xi_i^n\right)^2 < \frac{\xi}{12}, \text{ if } G \geq G_0,\ n \geq N_1, \tag{3.38}$$

$$\frac{3}{(1-\beta)^2} \sum_{|i| \geq G} \left(\xi_i^n\right)^2 < \frac{\xi}{12}, \text{ if } G \geq G_1. \tag{3.39}$$

Hence, (3.38)–(3.39) hold for any $G \geq G_2 := \max\{G_0, G_1\}$.

Therefore, from (3.34), (3.35), (3.38), and (3.39), we obtain that (3.32) is estimated by

$$\sum_{|i| \geq G_2} \left| y_{\cdot,i}^{\epsilon_n}(r) \right|_{\mathbf{R}^m}^2 \leq \frac{\xi}{4}, \text{ if } n \geq N_1,$$

for $r \in](\hat{n}-1)\epsilon_n, \hat{n}\epsilon_n]$. In particular, this holds for t^*.

Now, taking $K_2 \geq \max\{G_2, K_1\}$ in (3.30), it follows the existence of $N(\xi) \geq N_1$ such that

$$\sum_{j=1}^{m} \sum_{|i| \leq K_2} \left| y_{j,i}^{\epsilon_n}\left(t^*\right) - w_{j,i} \right|^2 \leq \frac{\xi}{4}, \text{ if } n \geq N(\xi).$$

Hence, $\|y^{\epsilon_n}(t^*) - w\|_{l_m^2}^2 \leq \xi$ if $n \geq N$, and the precompacity of $\{y^{\epsilon_n}(t^*)\}$ follows.

Finally, applying the Ascoli–Arzelà theorem, we have the existence of a subsequence of $y_{\epsilon_n}(.)$ converging in $C\left(I; l_m^2\right)$ to some function $y(\cdot)$, where $I = [0, \hat{\alpha}]$.

It remains to check that $y(\cdot)$ is a solution. As every continuous function is uniformly continuous on a compact set [11, p.29], we have $F(y^{\epsilon_n}(t)) \to F(y(t))$ uniformly on $[0, \hat{\alpha}]$. Also, it is easy to see that $F(y^{\epsilon_n}(t-\epsilon_n)) \to F(y(t))$ uniformly on $[0, \hat{\alpha}]$. Hence, passing to the limit in the equality

$$y^{\epsilon_n}(t) = y^0 + \int_0^t F(y^{\epsilon_n}(s-\epsilon_n))\, ds$$

we obtain that $y(t) = y^0 + \int_0^t F(y(s))\, ds$, for all $t \in I$. □

To finish this section, we prove now a result which will allow us to extend the local solution to global ones.

Theorem 3.2. *Assume (H2), (H4). Then a solution* $y\,(t)$ *has a maximal interval of existence* $[0, T^*)$, *with* $T^* < \infty$, *if and only if* $\|y\,(t)\|_{l^2_m} \underset{t\to T^*}{\to} +\infty$.

Proof. We prove first that if $T^* < \infty$, then $\overline{lim}_{t\to T^*} \|y\,(t)\|_{l^2_m} = \infty$. By contradiction, let $T^* < \infty$ and let there exist K such that $\sup_{t\in[0,T^*)} \|y\,(t)\|_{l^2_m} \leq K$.

We can see that the limit $\lim_{t\to T^*} y\,(t)$ exists. Indeed, it follows from (3.7) that F is a bounded function. Then

$$y\,(t) = y^0 + \int_0^t F\,(y\,(s))\,ds,\ t \in \left[0, T^*\right),$$

is an uniformly continuous function on $[0, T^*)$. Applying the principle of extension by continuity [11, p.23, Theorem 17], we obtain that $y\,(t)$ can be extended to an uniformly continuous function on $[0, T^*]$. Denoting $y^* = \lim_{t\to T^*} y\,(t)$, it is easy to see that the new function

$$\hat{y}\,(t) := \begin{cases} y\,(t), & \text{if } t \in [0, T^*), \\ y^*, & \text{if } t = T^*, \end{cases}$$

is a solution on $[0, T^*]$, which contradicts the assumption of maximality.

Thus, if $T^* < \infty$, then $\overline{lim}_{t\to T^*} \|y\,(t)\|_{l^2_m} = \infty$.

We prove again by contradiction that $\underline{lim}_{t\to T^*} \|y\,(t)\|_{l^2_m} = \infty$. Suppose the existence of $t_n \nearrow T^*$ and K such that $\|y\,(t_n)\|_{l^2_m} \leq K$. The continuity of the map $t \mapsto \|y\,(t)\|_{l^2_m}$ and $\overline{lim}_{t\to\infty} \|y\,(t)\|_{l^2_m} = \infty$ implies the existence of $h_n \to 0$ satisfying

$$\begin{aligned} \|y\,(t)\|_{l^2_m} &\leq K + 1,\ \text{if } t_n \leq t \leq t_n + h_n, \\ \|y\,(t_n + h_n)\|_{l^2_m} &= K + 1. \end{aligned}$$

Thus,

$$\begin{aligned} K + 1 = \|y\,(t_n + h_n)\|_{l^2_m} &= \left\| y_0 + \textstyle\int_0^{t_n+h_n} F\,(y\,(s))\,ds \right\|_{l^2_m} \\ &\leq \|y\,(t_n)\|_{l^2_m} + \left\| \textstyle\int_{t_n}^{t_n+h_n} F\,(y\,(s))\,ds \right\|_{l^2_m} \leq K + \textstyle\int_{t_n}^{t_n+h_n} \|F\,(y\,(s))\|_{l^2_m}\,ds. \end{aligned}$$

Since there exists $M > 0$ such that $\|F\,(y\,(s))\|_{l^2_m} \leq M$, for all $t_n \leq s \leq t_n + h_n$, we have

$$K + 1 = \|y\,(t_n + h_n)\|_{l^2_m} \leq K + h_n M.$$

As $h_n \to 0$, for $0 < \epsilon < 1$, there exists $N\,(\epsilon)$ such that $h_n M < \epsilon$ if $n > N$. Then

$$K + 1 = \|y\,(t_n + h_n)\|_{l^2_m} \leq K + \epsilon,\ \forall n > N,$$

which is a contradiction. □

From this theorem, we deduce that if a local solution is uniformly bounded with respect to t, then it can be extended to a global one. We remark here that conditions $(H1)$–$(H4)$ do not provide uniqueness of the Cauchy problem, as we will show at the end of Sect. 3.3 with an example.

3.2 A Priori Estimates

In this section, we shall obtain some estimates which are necessary in order to prove the existence of the global attractor. On the one hand, we prove the existence of an absorbing bounded set. On the other hand, we estimate the tails of the solutions.

3.2.1 Existence of a Bounded Absorbing Set

We note that from the integral representation of the solutions, it follows that any solution $y(\cdot)$ satisfies $y(\cdot) \in C^1\left([0,T^*);l_m^2\right)$, where T^* is the maximal time of existence. Then

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|_{l_m^2}^2 = \left(\dot{y}, y\right)_{l_m^2}, \ \forall t \in \left(0, T^*\right).$$

Proposition 3.2. *Assume (H1)–(H4). Then every solution u verifies*

$$\|u(t)\|_{l_m^2}^2 \le \|u(0)\|_{l_m^2}^2 e^{-2\alpha t} + \frac{\|c_0\|_{l^1}}{\alpha}, \ \forall t \in \left[0, T^*\right[. \tag{3.40}$$

Proof. We multiply (3.3) by u. Then using (H1) and (H3), we have

$$\begin{aligned}
\left(\dot{u}, u\right)_{l_m^2} &= -(aAu, u)_{l_m^2} - \left(\hat{f}(u), u\right)_{l_m^2} \\
&= -\sum_{i\in\mathbf{Z}} \left(\tfrac{1}{2}(a+a^t)\left[\hat{B}u_{1,i} \ \ldots \ \hat{B}u_{m,i}\right], \left[\hat{B}u_{1,i} \ \ldots \ \hat{B}u_{1,i}\right]\right)_{\mathbf{R}^m} \\
&\quad - \left(\hat{f}(u), u\right)_{l_m^2} \\
&\le -\sum_{i\in\mathbf{Z}} \beta \|Bu_{\cdot,i}\|_{\mathbf{R}^m}^2 - \alpha \sum_{i\in\mathbf{Z}} |u_{\cdot,i}|_{\mathbf{R}^m}^2 + \sum_{i\in\mathbf{Z}} c_{0,i}\,.
\end{aligned}$$

Hence,

$$\frac{1}{2}\frac{d}{dt}\|u\|_{l_m^2}^2 + \alpha\|u\|_{l_m^2}^2 \le \|c_0\|_{l^1}\,.$$

By the Gronwall's lemma, we obtain (3.40). □

Now, Theorem 3.2 and Proposition 3.2 imply the following consequences:

Corollary 3.1. *Every local solution can be extended to a global one and (3.40) is true for all* $t \in [0, \infty)$.

Corollary 3.2. *The bounded set defined by*

$$M := \left\{ u \in l_m^2 : \|u\|_{l_m^2} \leq R_0 \right\}, \tag{3.41}$$

where $R_0^2 := 1 + \frac{\|c_0\|_{l^1}}{\alpha}$, *is absorbing, that is, for any bounded set* $B \subset l_m^2$, *there exists* $T(B)$ *such that for any solution with* $u_0 \in B$, *we have* $\|u(t)\|_{l_m^2} \leq R_0, \ \forall t \geq T(B)$.

3.2.2 Estimate of the Tails

Further, we prove an uniform estimate of the tails of solutions.

Lemma 3.4. *Assume (H1)–(H4) and let* B *be a bounded set of* l_m^2. *For any* $\epsilon > 0$, *there exist* $T(\epsilon, B), K(\epsilon, B) > 0$ *such that for every solution with* $u_0 \in B$, *it holds*

$$\sum_{|i| \geq 2K(\epsilon, B)} |u_{\cdot,i}(t)|_{\mathbf{R}^m}^2 \leq \epsilon, \ \forall\, t \geq T(\epsilon, B). \tag{3.42}$$

Proof. Define a smooth function θ satisfying

$$\theta(s) = \begin{cases} 0, \ 0 \leq s \leq 1, \\ 0 \leq \theta(s) \leq 1, \ 1 \leq s \leq 2, \\ 1, \ s \geq 2. \end{cases}$$

Obviously, $|\theta'(s)| \leq C$, for all $s \in \mathbf{R}^+$. For any solution u, define $v := (v_{\cdot,i})_{i \in \mathbf{Z}}$ by $v_{\cdot,i} = \rho_{K,i} u_{\cdot,i}$, where $\rho_{K,i} := \theta\left(\frac{|i|}{K}\right)$.

We note that $u \in C^1\left([0, \infty); l_m^2\right)$ implies the equality

$$\frac{1}{2}\frac{d}{dt}\sum_{i \in \mathbf{Z}} \rho_{K,i} |u_{\cdot,i}|_{\mathbf{R}^m}^2 = \sum_{i \in \mathbf{Z}} \left(\frac{d}{dt}u_{\cdot,i}, v_{\cdot,i}\right)_{\mathbf{R}^m}, \ \forall t > 0.$$

Multiplying (3.3) by $v_{\cdot,i}$ and summing over $\mathbf{Z}$, we obtain

$$\frac{1}{2}\frac{d}{dt}\sum_{i \in \mathbf{Z}} \rho_{K,i} |u_{\cdot,i}|_{\mathbf{R}^m}^2 + \sum_{i \in \mathbf{Z}} (aBu_{\cdot,i}, Bv_{\cdot,i})_{\mathbf{R}^m} = -\sum_{i \in \mathbf{Z}} \rho_{K,i} (f(u_{\cdot,i}), u_{\cdot,i})_{\mathbf{R}^m}. \tag{3.43}$$

Using (H3), we get

$$
\begin{aligned}
\left(a\,(Bu)_{\cdot,i}\,,(Bv)_{\cdot,i}\right)_{\mathbf{R}^m} &= \rho_{K,i}\,\left(a\,[u_{\cdot,i+1}-u_{\cdot,i}],[u_{\cdot,i+1}-u_{\cdot,i}]\right)_{\mathbf{R}^m}\\
&\quad+\left(a\,[u_{\cdot,i+1}-u_{\cdot,i}],[\rho_{K,i+1}-\rho_{K,i}]\,u_{\cdot,i+1}\right)_{\mathbf{R}^m}\\
&= \rho_{K,i}\left(\frac{1}{2}\left(a+a^t\right)[u_{\cdot,i+1}-u_{\cdot,i}],[u_{\cdot,i+1}-u_{\cdot,i}]\right)_{\mathbf{R}^m}\\
&\quad+\left(a\,[u_{\cdot,i+1}-u_{\cdot,i}],[\rho_{K,i+1}-\rho_{K,i}]\,u_{\cdot,i+1}\right)_{\mathbf{R}^m}\\
&\geq \rho_{K,i}\,\beta\,\|u_{\cdot,i+1}-u_{\cdot,i}\|^2_{\mathbf{R}^m}+\left(\rho_{K,i+1}-\rho_{K,i}\right)\\
&\quad\times\left[(au_{\cdot,i+1},u_{\cdot,i+1})_{\mathbf{R}^m}-(au_{\cdot,i}\,,u_{\cdot,i+1})_{\mathbf{R}^m}\right].
\end{aligned}
\tag{3.44}
$$

In view of Proposition 3.2, the sum of the last two terms in (3.44) can be estimated as follows:

$$
\begin{aligned}
\sum_{i\in\mathbf{Z}}\left|\left(\rho_{K,i+1}-\rho_{K,i}\right)\left(au_{\cdot,i}\,,u_{\cdot,i+1}\right)_{\mathbf{R}^m}\right| &\leq \sum_{i\in\mathbf{Z}}\sum_{j=1}^{m}\sum_{k=1}^{m}\left|\left(\rho_{K,i+1}-\rho_{K,i}\right)a_{jk}u_{k,i}u_{j,i+1}\right|\\
&\leq \max_{1\leq j,k\leq m}\left|a_{j,k}\right|\frac{1}{2}\sum_{i\in\mathbf{Z}}\left|\rho_{K,i+1}-\rho_{K,i}\right|\sum_{j=1}^{m}\sum_{k=1}^{m}\\
&\quad\left(\left|u_{k,i}\right|^2+\left|u_{j,i+1}\right|^2\right)\\
&\leq \widetilde{a}\frac{m}{2}\sum_{i\in\mathbf{Z}}\frac{\left|\theta'(\xi_i)\right|}{K}\left(\|u_{\cdot,i}\|^2_{\mathbf{R}^m}+\|u_{\cdot,i+1}\|^2_{\mathbf{R}^m}\right)\\
&\leq \widetilde{a}\frac{mC_1}{2K},
\end{aligned}
$$

$$
\sum_{i\in\mathbf{Z}}\left|\left(\rho_{K,i+1}-\rho_{K,i}\right)\right|\left(au_{\cdot,i+1},u_{\cdot,i+1}\right)_{\mathbf{R}^m}\leq\widetilde{a}m\sum_{i\in\mathbf{Z}}\frac{\left|\theta'(\xi_i)\right|}{K}\|u_{\cdot,i+1}\|^2_{\mathbf{R}^m}\leq\widetilde{a}\frac{mC_2}{K},
$$

where $\widetilde{a} := \max_{1\leq j,k\leq m}\left|a_{j,k}\right|$ and C_i are constants which depend on B. Therefore, from (3.44), we have

$$
\sum_{i\in\mathbf{Z}}\left(aBu_{\cdot,i}\,,Bv_{\cdot,i}\right)_{\mathbf{R}^m}\geq\sum_{i\in\mathbf{Z}}\rho_{K,i}\,\beta\,\|u_{\cdot,i+1}-u_{\cdot,i}\|^2_{\mathbf{R}^m}-\frac{C}{K},
$$

where C depends on B and the parameters of the problem.

Using (3.43) and $(H1)$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{i\in\mathbf{Z}}\rho_{K,i}\,|u_{\cdot,i}|^2_{\mathbf{R}^m}+\sum_{i\in\mathbf{Z}}\rho_{K,i}\,\beta\,\|u_{\cdot,i+1}-u_{\cdot,i}\|^2_{\mathbf{R}^m}\\
&\quad\leq\sum_{i\in\mathbf{Z}}\rho_{K,i}\left(c_{0,i}-\alpha\,|u_{\cdot,i}|^2_{\mathbf{R}^m}\right)+\frac{C}{K},
\end{aligned}
$$

so, in particular,

$$\frac{d}{dt}\sum_{i\in\mathbf{Z}}\rho_{K,i}\left|u_{\cdot,i}\right|^2_{\mathbf{R}^m}+2\alpha\sum_{i\in\mathbf{Z}}\rho_{K,i}\left|u_{\cdot,i}\right|^2_{\mathbf{R}^m}\leq 2\left\|\rho_{K,i}c_{0,i}\right\|_{l^1}+2\frac{C}{K}. \tag{3.45}$$

Since $c_0 \in l^1$, for any $\epsilon > 0$, there exists $K(\epsilon, B)$ such that $\|\rho_{K,i}c_{0,i}\|_{l^1} < \frac{\epsilon\alpha}{4}$ and $\frac{C}{K} < \frac{\epsilon\alpha}{4}$. Then, applying the Gronwall's lemma in (3.45), we have

$$\sum_{i\in\mathbf{Z}}\rho_{K,i}\left|u_{\cdot,i}(t)\right|^2_{\mathbf{R}^m}\leq e^{-2\alpha t}\sum_{i\in\mathbf{Z}}\rho_{K,i}\left|u_{\cdot,i}(0)\right|^2_{\mathbf{R}^m}+\frac{\epsilon}{2}.$$

It follows the existence of $T(\epsilon, B)$ such that

$$\sum_{|i|\geq 2K}\left|u_{\cdot,i}(t)\right|^2_{\mathbf{R}^m}\leq\sum_{i\in\mathbf{Z}}\rho_{K,i}\left|u_{\cdot,i}(t)\right|^2_{\mathbf{R}^m}\leq\epsilon,\ \forall t\geq T(\epsilon,B).$$

□

3.3 Existence of the Global Attractor

We start this section by defining the multivalued semiflow associated to the solutions of system (3.3). After that, we prove some properties of the semiflow like the upper semicontinuity with respect to the initial data or the asymptotic compactness. Using these properties jointly with those of the previous section, we establish the existence of a global compact invariant attractor.

Let $u^0 \in l^2_m$. Denote by $P\left(l^2_m\right)$ the set of all nonempty subsets of l^2_m. We set

$$\mathscr{D}\left(u^0\right)=\{u(\cdot):\ u \text{ is a solution of } (3.3) \text{ defined for any } t\geq 0\}.$$

We know that under conditions (H1)–(H4), this set is nonempty. Moreover, since Corollary 3.1 implies that every local solution can be extended to a global one, $\mathscr{D}\left(u^0\right)$ contains all the possible solutions corresponding to the initial data u_0.

The associated semiflow $G : \mathbf{R}^+ \times l^2_m \to P\left(l^2_m\right)$ is defined by

$$G(t,u^0)=\{z\in l^2_m:\exists u\in\mathscr{D}\left(u^0\right) \text{ such that } u(0)=u^0 \text{ and } u(t)=z\}.$$

Observe that this map is, in general multivalued, as conditions (H1)–(H4) do not guarantee the uniqueness of the Cauchy problem.

Lemma 3.5. *Assume (H1)–(H4). Then $G(t, G(s, x)) = G(t + s, x)$ for all $x \in l^2_m$, $s, t \in \mathbf{R}^+$, that is, G is an strict multivalued semiflow.*

Proof. First, let us show that $G(t+s,x) \subset G(t,G(s,x))$. Consider $y \in G(t+s,x)$. Then there exists $u(\cdot) \in \mathscr{D}(x)$ verifying $u(0) = x$ and $u(t+s) = y$. As $u(s) \in G(s,x)$, the result follows if we prove that $y \in G(t,u(s))$. Define $\overline{u}(\cdot) = u(\cdot + s)$. Using the integral representation of the solutions, we can easily see that $\overline{u}$ is a solution. Thus, $\overline{u}(t) = u(t+s) = y$, $\overline{u}(0) = u(s)$ and then $y \in G(t,u(s)) \subset G(t,G(s,x))$.

Now, we must prove that $G(t,G(s,x)) \subset G(t+s,x)$. Let $y \in G(t,G(s,x))$. Then there exist z_1, $u_1(\cdot) \in \mathscr{D}(x)$, and $u_2(\cdot) \in \mathscr{D}(z_1)$, such that

$$u_1(0) = x,\ u_1(s) = z_1,$$
$$u_2(0) = z_1,\ u_2(t) = y.$$

Defining

$$u(r) = \begin{cases} u_1(r), \text{ if } 0 \le r \le s, \\ u_2(r-s), \text{ if } s \le r. \end{cases}$$

we obtain a new solution (this is proved using again the integral representation), so that $u(\cdot) \in \mathscr{D}(u_0)$ and $u(0) = x,\ u(t+s) = y$. It follows that $y \in G(t+s,x)$. □

Let us prove further some properties of the multivalued semiflow G.

Lemma 3.6. *Suppose that (H1)–(H4) hold. Let u^{0n} be a sequence converging to u^0 in l_m^2 and fix $T > 0$. Then for any $\epsilon > 0$, there exists $K(\epsilon)$ such that for any solution $u^n(\cdot) \in \mathscr{D}(u^{0n})$, we have*

$$\sum_{|i| \ge K(\epsilon)} \left|u_{\cdot,i}^n(t)\right|_{\mathbf{R}^m}^2 \le \epsilon,\ \forall\, t \in [0,T]. \tag{3.46}$$

Moreover, there exists $u(\cdot) \in \mathscr{D}(u^0)$ and a subsequence u^{n_k} satisfying

$$u^{n_k} \to u \text{ in } C\left([0,T], l_m^2\right). \tag{3.47}$$

Proof. Define $v = (v_{\cdot,i})_{i \in \mathbf{Z}}$ by $v_{\cdot,i} = \rho_{K,i} u_{\cdot,i}$. Clearly, for any $\epsilon > 0$, there exist $K_1(\epsilon)$, $N(\epsilon)$ such that

$$\textstyle\sum_{i \in \mathbf{Z}} \left|u_{\cdot,i}^{0n} - u_{\cdot,i}^0\right|_{\mathbf{R}^m}^2 < \frac{\epsilon}{4},\ \forall n \ge N,$$
$$\textstyle\sum_{i \in \mathbf{Z}} \rho_{K,i} \left|u_{\cdot,i}^0\right|_{\mathbf{R}^m}^2 < \frac{\epsilon}{4},\ \forall K \ge K_1,$$

so

$$\sum_{i \in \mathbf{Z}} \rho_{K,i} \left|u_{\cdot,i}^{0n}\right|_{\mathbf{R}^m}^2 \le 2\left(\sum_{i \in \mathbf{Z}} \rho_{K,i} \left|u_{\cdot,i}^{0n} - u_{\cdot,i}^0\right|_{\mathbf{R}^m}^2 + \sum_{i \in \mathbf{Z}} \rho_{K,i} \left|u_{\cdot,i}^0\right|_{\mathbf{R}^m}^2\right) < \epsilon, \tag{3.48}$$

if $n \ge N$ and $K \ge K_1$. Also, in view of Proposition 3.2, there exists $R_0 > 0$ verifying

$$\|u^n(t)\|_{l_m^2} \le R_0, \forall t \in [0,T], \forall n. \tag{3.49}$$

Using (3.45) and $c_0 \in l^1$, one can find $K_2(\epsilon)$ such that

$$\frac{d}{dt}\sum_{i\in\mathbf{Z}} \rho_{K,i} |u_i^n|^2_{\mathbf{R}^m} \le \epsilon, \text{ if } K \ge K_2.$$

Integrating over $(0,t)$ and using (3.48), we obtain

$$\sum_{|i|\ge 2K(\epsilon)} |u_i^n(t)|^2_{\mathbf{R}^m} \le \sum_{i\in\mathbf{Z}} \rho_{K,i} |u_i^n(t)|^2_{\mathbf{R}^m} \le \epsilon + T\epsilon, \text{ if } K \ge \max\{K_1, K_2\}, \tag{3.50}$$

so that (3.46) holds.

Fix now $t \in [0,T]$. In view of (3.49), passing to a subsequence, we can state that $u^n(t) \to w$ weakly in l_m^2. Then for any $\sigma > 0$, there exist $\widetilde{N}(\sigma)$ and $\widetilde{K}(\sigma)$ such that

$$\begin{aligned}\|u^n(t) - w\|_{l_m^2} &\le \sum_{|i|\le \widetilde{K}(\sigma)} \left|u^n_{\cdot,i}(t) - w_{\cdot,i}\right|^2_{\mathbf{R}^m} + \sum_{|i|> \widetilde{K}(\sigma)} \left|u^n_{\cdot,i}(t) - w_{\cdot,i}\right|^2_{\mathbf{R}^m} \\ &\le \sum_{|i|\le \widetilde{K}(\sigma)} \left|u^n_{\cdot,i}(t) - w_{\cdot,i}\right|^2_{\mathbf{R}^m} + 2\sum_{|i|> \widetilde{K}(\sigma)} \left|u^n_{\cdot,i}(t)\right|^2_{\mathbf{R}^m} \\ &\quad + 2\sum_{|i|> \widetilde{K}(\sigma)} |w_{\cdot,i}(t)|^2_{\mathbf{R}^m} < \sigma,\end{aligned}$$

if $n \ge \widetilde{N}$. Hence, $u^n(t) \to w$ strongly in l_m^2. Therefore, the sequence $u^n(t)$ is precompact for any t. By (3.7) and (3.49),

$$\|F(u^n(t))\|^2_{l_m^2} \le C, \ \forall n \in \mathbf{N},$$

and from the equality $\frac{d}{dt}u^n_{\cdot,i} = F_{\cdot,i}\left(u^n_{\cdot,i}\right)$, we obtain that

$$\left\|\frac{d}{dt}u^n(t)\right\|_{l_m^2} \le C_1,$$

proving that the sequence u^n is equicontinuous. The Ascoli–Arzelà theorem implies then the existence of a subsequence u^{n_k} converging in $C\left([0,T], l_m^2\right)$ to some function $u(\cdot)$. Arguing as in the proof of Theorem 3.1, we obtain that u is a solution. Also, it is clear that $u(0) = u^0$. □

As a consequence of this lemma, we obtain that the graph of $G(t,\cdot)$ is closed and also that the map G has compact values.

Corollary 3.3. *Assume (H1)–(H4). Then the graph of $G(t,.)$ is closed, that is, if $\xi^n \to \xi^\infty$ and $\beta^n \to \beta^\infty$, in l_m^2, where $\xi^n \in G(t,\beta^n)$, then $\xi^\infty \in G(t,\beta^\infty)$.*

Proof. For $\xi^n \in G(t,\beta^n)$, there are $u^n(.) \in \mathscr{D}(\beta^n)$ such that $u^n(t) = \xi^n$. Applying Lemma 3.6, we obtain, up to a subsequence, that $u^n \to u$ in $C\left([0,t]; l_m^2\right)$, where $u(\cdot) \in D(\beta^\infty)$. In particular, $\xi^n = u^n(t) \to u(t) = \xi^\infty \in G(t,\beta^\infty)$. □

Corollary 3.4. *Assume (H1)–(H4). Then the map $G(t,\cdot)$ has compact values.*

Proof. Let $\xi_n \in G\left(t,x^0\right)$ be an arbitrary sequence. Then there exist $u^n(\cdot) \in \mathscr{D}\left(x^0\right)$ such that $u^n(0) = x^0$, $u^n(t) = \xi_n$. In view of (3.47), there exists $u \in D\left(x^0\right)$ satisfying $u^{n_k} \to u$ in $C\left([0,t]; l_m^2\right)$ for some subsequence. Hence, $\xi_{n_k} = u^{n_k}(t) \to u(t) = \xi \in G\left(t,x^0\right)$ in l_m^2. □

Recall that the multivalued map $G(t,\cdot) : l_m^2 \to P\left(l_m^2\right)$ is called upper semicontinuous if for all $x_0 \in l_m^2$ and any neighborhood $O(G(t,x_0))$, there exists $\delta > 0$ such that $G(t,x) \subset O(G(t,x_0))$, as soon as $\|x - x_0\|_{l_m^2} < \delta$.

Proposition 3.3. *Assume (H1)–(H4). Then the map $G(t,.)$ is upper semicontinuous for all $t \geq 0$.*

Proof. Suppose the opposite. Then there exist x_0, $t > 0$, a neighborhood $O(G(t,x_0))$, and sequences $x_n \to x_0$, $y_n \in G(t,x_n)$ such that $y_n \notin O(G(t,x_0))$. Let $y_n = u_n(t)$, where $u_n \in \mathscr{D}(x_n)$ are such that $u^n(0) = x_n$. By (3.47), we obtain that, up to a subsequence, $u^n(\cdot) \to u(\cdot) \in \mathscr{D}(x_0)$ in $C\left([0,t]; l_m^2\right)$. In particular, $y_n = u^n(t) \to u(t) \in G(t,x_0)$ in l_m^2, which is a contradiction. □

Now, we can prove the asymptotic compactness of the semiflow.

By $\mathscr{B}\left(l_m^2\right)$, we shall denote the set of all nonempty bounded subsets of l_m^2. Also, let $\gamma_T^+(B) := \cup_{t \geq T} G(t,B)$.

Lemma 3.7. *Assume (H1)–(H4). Then G is asymptotically upper semicompact, that is, for any $B \in \mathscr{B}\left(l_m^2\right)$ such that there exists $T(B) \in \mathbf{R}^+$ verifying $\gamma_{T(B)}^+ \in \mathscr{B}\left(l_m^2\right)$, any sequence $\xi^n \in G(t_n,B)$, $t_n \to \infty$, is precompact in l_m^2.*

Proof. Observe that there exists n_0 such that $t_n \geq T(B)$ if $n \geq n_0$. Then, $\{\xi^n\}_{n\in\mathbf{N}}$, where $\xi^n \in G(t_n,B)$, is bounded in l_m^2, so that passing to a subsequence, we obtain

$$\xi^n \to \xi \ \text{ weakly in } l_m^2. \tag{3.51}$$

We shall check that the convergence is in fact strong. Using (3.42), we obtain that for any $\epsilon > 0$, there exist $n_1(\epsilon), K_0(\epsilon)$ such that $\sum_{|i|>K_0} |\xi^n|_{\mathbf{R}^m}^2 \leq \epsilon$, if $n \geq n_1$. Also, as $\xi \in l_m^2$, one can find $K_1(\epsilon) \geq K_0$ such that for $K \geq K_1$, $\sum_{|i|>K} |\xi_{\cdot,i}|_{\mathbf{R}^m}^2 \leq \epsilon$. Then

$$\begin{aligned}\sum_{i\in\mathbf{Z}} \left|\xi_{\cdot,i}^n - \xi_{\cdot,i}\right|_{\mathbf{R}^m}^2 &= \sum_{|i|\leq K_1} \left|\xi_{\cdot,i}^n - \xi_{\cdot,i}\right|_{\mathbf{R}^m}^2 + \sum_{|i|>K_1} \left|\xi_{\cdot,i}^n - \xi_{\cdot,i}\right|_{\mathbf{R}^m}^2 \\ &\leq \epsilon + 2\sum_{|i|>K_1} \left|\xi_{\cdot,i}^n\right|_{\mathbf{R}^m}^2 + 2\sum_{|i|>K_1} |\xi_{\cdot,i}|_{\mathbf{R}^m}^2 \leq 5\epsilon,\end{aligned}$$

if $n \geq n_2(\epsilon) \geq n_1$.

Thus, $\{\xi^n\}$ is precompact in l_m^2. □

We recall (see Chap. 1) that the set $\mathscr{A}$ is said to be a global attractor of G if:

1. It is negatively semiinvariant (that is, $A \subset G(t, A)$, for all $t \geq 0$).
2. It is attracting, that is,

$$dist\,(G(t, B), \mathscr{A}) \to 0, \text{ as } t \to +\infty, \tag{3.52}$$

for all B bounded in l_m^2, where $dist\,(C, A) = \sup_{c \in C} \inf_{a \in A} \|c - a\|_{l_m^2}$ is the Hausdorff semidistance.

The global attractor is said to be invariant if $\mathscr{A} = G(t, \mathscr{A})$, for all $t \geq 0$. It is called minimal if for any closed set Y satisfying (3.52), we have $\mathscr{A} \subset Y$. We note that if the global attractor is bounded, then it is minimal (see Remark 1.5).

Theorem 3.3. *Assume (H1)–(H4). Then system (3.3) defines a multivalued semiflow in the phase space l_m^2 which possesses a global compact attractor $\mathscr{A}$. Moreover, it is minimal and invariant.*

Proof. We have proved the following properties of G :

1. In view of Proposition 3.2 and Corollary 3.2, $\gamma_0^+(B) = \cup_{t \geq 0} G(t, B)$ is bounded for any bounded set B and a bounded absorbing set B_0 exists.
2. Lemma 3.7 implies that G is asymptotically upper semicompact.
3. From Proposition 3.3 and Corollary 3.4, it follows that G is upper semicontinuous and has compact values.
4. By Lemma 3.5 G, is a strict multivalued semiflow.

Then the result follows from Theorem 1.3 and Remark 1.7. □

To finish this section, we shall consider an example of a lattice system where we have more than one solution corresponding to a given initial data, so that the system is really multivalued.

Let us consider system (3.2) in the scalar case with $a = 1$, that is,

$$\begin{cases} \dot{u}_i = \frac{1}{h^2}(u_{i-1} - 2u_i + u_{i+1}) - f_i(u_i), \\ u_i(0) = \left(u^0\right)_i. \end{cases} \tag{3.53}$$

We define $v_i = \frac{1}{2^{|i|}}$. Then $v = (v_i)_{i \in \mathbf{Z}} \in l^2$ and

$$v_{i-1} - 2v_i + v_{i+1} = \frac{1}{2^{i-1}} - \frac{2}{2^i} + \frac{1}{2^{i+1}} = \frac{1}{2}v_i, \text{ if } i > 0,$$
$$v_{i-1} - 2v_i + v_{i+1} = \frac{1}{2^{-i-1}} - \frac{2}{2^{-i}} + \frac{1}{2^{-i+1}} = \frac{1}{2}v_i, \text{ if } i < 0,$$
$$v_{i-1} - 2v_i + v_{i+1} = \frac{1}{2} - 2 + \frac{1}{2} = -1 = -v_0, \text{ if } i = 0.$$

Let $f = (f_i)_{i\in\mathbf{Z}}$, $f_i : \mathbf{R} \to \mathbf{R}$ be given by

$$f_0(u_0) = \begin{cases} -\frac{u_0}{h^2} - \sqrt{u_0},\ u_0 \in [0,1], \\ -\frac{u_0}{h^2} + (u_0 - 2)\,u_0^2,\ u_0 \notin [0,1], \end{cases}$$

$$f_i(u_i) = \frac{u_i}{2h^2} - \frac{\sqrt{|u_i|}}{2^{\frac{|i|}{2}}}, \text{ if } i \neq 0.$$

Let us prove that conditions $(H1)$–$(H4)$ hold. $(H3)$–$(H4)$ are obvious. For $(H1)$, we have

$$f_i(u_i)\,u_i = \frac{1}{2h^2}u_i^2 - \frac{\sqrt{|u_i|}u_i}{2^{\frac{|i|}{2}}} \geq \frac{1}{4h^2}u_i^2 - \frac{K_{0h}}{2^{2|i|}}, \text{ if } i \neq 0,$$

$$f_0(u_0)\,u_0 = -\frac{u_0^2}{h^2} + (u_0 - 2)\,u_0^3 \geq \frac{1}{2}u_0^4 - K_{1h} \geq u_0^2 - K_{2h}, \text{ if } u_0 \notin [0,1],$$

$$f_0(u_0)\,u_0 = -\frac{u_0^2}{h^2} - \sqrt{u_0}u_0 \geq u_0^2 - K_{3h}, \text{ if } u_0 \in [0,1].$$

Then $(H1)$ holds with $c_{0,i} = \frac{K_{4h}}{2^{2|i|}}$, $K_{4h} = \max\{K_{0h}, K_{2h}, K_{3h}\}$, $\alpha = \min\{\frac{1}{4h^2}, 1\}$. Condition $(H2)$ follows from

$$|f_i(u_i)| \leq \frac{1}{2h^2}|u_i| + \frac{|u_i|}{2} + \frac{1}{2^{|i|+1}} \leq K_{5h}|u_i| + \frac{1}{2^{|i|+1}}, \text{ if } i \neq 0,$$

$$|f_0(u_0)| \leq \frac{|u_0|}{h^2} + |u_0|^3 + 2u_0^2 \leq K_h(|u_0|)\,|u_0|, \text{ if } u_0 \notin [0,1],$$

$$|f_0(u_0)| \leq \frac{|u_0|}{h^2} + \sqrt{|u_0|} \leq K_{6h}|u_0| + \frac{1}{2}, \text{ if } u_0 \in [0,1],$$

where $K_h(x)$ is continuous and increasing. $(H1)$ holds with

$$C(|x|) = \max\{K_{5h}, K_h(|x|), K_{6h}\}$$

and $c_{1,i} = \frac{1}{2^{|i|+1}}$.

Let us show that there exist at least two solutions corresponding to the initial data $u^0 = 0$, that is, $\left(u^0\right)_i = 0$ for any i.

Obviously, $u^1(t) \equiv 0$ is a solution. Define also

$$u(t) = \begin{cases} \frac{t^2}{4}v, \text{ if } 0 \leq t \leq 2, \\ w(t-2), \text{ if } t \geq 2, \end{cases}$$

where $v = (v_i)_{i\in\mathbf{Z}} \in l^2$ is given by $v_i = \frac{1}{2^{|i|}}$ and $w(t)$ is a solution of the problem with initial data $w(0) = v$.

As the concatenation of solutions is again a solution (see the proof of Lemma 3.5), we need to check only that u is a solution on $[0,2]$. We note that

$$\begin{aligned} \frac{d}{dt}u_i &= \frac{t}{2}\frac{1}{2^{|i|}}, \\ \frac{u_{-1}-2u_0+u_1}{h^2} - f_0\left(u_0\right) &= -\frac{t^2}{4h^2} + \left(\frac{t^2}{4h^2} + \frac{t}{2}\right) = \frac{t}{2}, \\ \frac{u_{i-1}-2u_i+u_{i+1}}{h^2} - f_i\left(u_i\right) &= \frac{1}{2h^2 2^{|i|}} - \frac{1}{2h^2 2^{|i|}} + \frac{t}{2}\frac{1}{2^{|i|}} = \frac{t}{2}\frac{1}{2^{|i|}}, \text{ if } i \neq 0. \end{aligned}$$

Hence, $u\left(t\right)$ is a solution.

We can apply Theorem 3.3, and then the multivalued semiflow generated by (3.53) possesses a global compact attractor $\mathscr{A}$. Also, let us show that the dynamics inside the attractor is multivalued.

We recall that the point $\bar{x} \in l^2$ is called a stationary point of the multivalued semiflow G, if $\bar{x} \in G(t,\bar{x})$, for all $t \in \mathbf{R}^+$. The map $\varphi : \mathbf{R} \mapsto l^2$ is called a complete trajectory of the multivalued semiflow G if

$$\varphi(t+s) \in G(t,\varphi(s)),\ \forall\ s \in \mathbf{R},\ \forall\ t \in \mathbf{R}^+.$$

We need the following lemma.

Lemma 3.8. *Let the multivalued semiflow $G : \mathbf{R}_+ \times X \to P\left(X\right)$ satisfies the conditions of Theorem 1.2, so that G possesses a global attractor $\mathscr{A}$. If $\bar{x}$ is a stationary point, then $\bar{x} \in \mathscr{A}$. If $\varphi : \mathbf{R} \mapsto X$ is a bounded complete trajectory, then $\bigcup_{s\in\mathbf{R}} \varphi(s) \subset \mathscr{A}$.*

Proof. In view of the attraction property for any $\epsilon > 0$, there exists $T > 0$ such that

$$G(t,\bar{x}) \subset O_\epsilon(\omega(\bar{x})),\ \forall\ t \geq T.$$

Then for any $t \geq T$,

$$\bar{x} \in G(t,\bar{x}) \subset O_\epsilon(\omega(\bar{x})),$$

and then, $\bar{x} \in cl_X\omega(\bar{x}) = \omega(\bar{x}) \subset \mathscr{A}$.

Let $\varphi : \mathbf{R} \mapsto X$ be a bounded complete trajectory, that is, $B := \bigcup_{s\in\mathbf{R}} \varphi(s) \in \beta(X)$. Then for any $\epsilon > 0$, there exists $T > 0$ such that

$$G(t,\bigcup_{s\in\mathbf{R}}\varphi(s)) \subset O_\epsilon(\omega(B)),\ \forall\ t \geq T.$$

Hence, for any $s \in \mathbf{R}$, $t \geq T$,

$$\varphi(s) \in G(t,\varphi(s-t)) \subset G(t,\bigcup_{s\in\mathbf{R}}\varphi(s)) \subset O_\epsilon(\omega(B)),$$

and then, $\varphi(s) \in \omega(B) \subset \mathscr{A}$. □

We note that 0 is a stationary point of G. Hence, it belongs to $\mathscr{A}$. We define also the function

$$u(t) = \begin{cases} 0, \ t \in (-\infty, 0), \\ \frac{1}{4}t^2 v, \ t \in [0,2], \\ w(t-2), \ t \geq 2, \end{cases}$$

where $w(t)$ is a solution of the problem with initial data $w(0) = v$. It follows from the proof of Lemma 3.5 and inequality (3.40) that $u(\cdot)$ is a bounded complete trajectory, so that it belongs to the global attractor.

Therefore, for the initial data 0, there exist at least two solutions which belong to the attractor.

3.4 Approximation of the Attractor

In this section, we shall consider finite-dimensional approximations of system (3.3). We prove the existence of a global compact attractor on each approximative system and after that the upper semicontinuity of these attractors with respect to the attractor of system (3.3).

Let $n \geq 1$. We shall consider the system of $(2n+1)^m$ ordinary differential equations given by

$$\begin{cases} \dot{v}_{\cdot,-n} = a\left(v_{\cdot,n} - 2v_{\cdot,-n} + v_{\cdot,-n+1}\right) - f_{\cdot,-n}\left(v_{\cdot,-n}\right), \\ \dot{v}_{\cdot,-n+1} = a\left(v_{\cdot,-n} - 2v_{\cdot,-n+1} + v_{\cdot,-n+2}\right) - f_{\cdot,-n+1}\left(v_{\cdot,-n+1}\right), \\ \qquad \cdots \\ \dot{v}_{\cdot,n-1} = a\left(v_{\cdot,n-2} - 2v_{\cdot,n-1} + v_{\cdot,n}\right) - f_{\cdot,n-1}\left(v_{\cdot,n-1}\right), \\ \dot{v}_{\cdot,n} = a\left(v_{\cdot,n-1} - 2v_{\cdot,n} + v_{\cdot,-n}\right) - f_{\cdot,n}\left(v_{\cdot,n}\right), \end{cases}$$

with initial data

$$\left(v_{\cdot,-n}, \ldots, v_{\cdot,n}\right)(0) = \left(v^0_{\cdot,-n}, \ldots, v^0_{\cdot,n}\right) \in \left[\mathbf{R}^{2n+1}\right]^m. \tag{3.54}$$

We note that the elements $v_{\cdot,-n-1}$ and $v_{\cdot,n+1}$, which are not defined, have been changed by $v_{\cdot,n}$ and $v_{\cdot,-n}$, respectively. This system can be rewritten as

$$\begin{cases} \dot{v} = -aA_n v - f_n(v), \ t > 0, \\ v(0) = v_0 \in \left[\mathbf{R}^{2n+1}\right]^m, \end{cases} \tag{3.55}$$

where $v = \left(v_{j,i}\right)_{|i| \leq n, j=1,\ldots,m}$, $f_n(v) = \left(f_{j,i}\left(v_{\cdot,i}\right)\right)_{|i| \leq n, j=1,\ldots,m}$, and the matrixes A_n, B_n (of order $(2n+1) \times (2n+1)$) are defined by

$$\hat{A}_n := \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & & 2 & -1 \\ -1 & 0 & 0 & \dots & -1 & 2 \end{bmatrix}, \hat{B}_n := \begin{bmatrix} -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & .. & -1 & 1 \\ 1 & 0 & 0 & \dots & 0 & -1 \end{bmatrix},$$

$$A_n = \left[\hat{A}_n \overset{(m)}{\dots} \hat{A}_n \right], \ B_n = \left[\hat{B}_n \overset{(m)}{\dots} \hat{B}_n \right].$$

One can check that $A_n = B_n B_n^t = B_n^t B_n$.

We know by Peano's existence theorem that for any initial condition in $\left[\mathbf{R}^{2n+1}\right]^m$, there exists at least one local solution. Further, we show that every local solution can be extended to a global one and that system (3.55) generates a multivalued semiflow G_n.

In the sequel, we denote by $\langle \cdot, \cdot \rangle$ the scalar product in $\left[\mathbf{R}^{2n+1}\right]^m$ given by

$$\langle u, v \rangle := \sum_{j=1}^{m} \sum_{|i| \leq n} u_{j,i} v_{j,i}, \ u, v \in \left[\mathbf{R}^{2n+1}\right]^m.$$

As in the case of the operator A (see Sect. 3.1), we can prove that

$$\langle a A_n u, v \rangle = \langle a B_n u, B_n v \rangle.$$

Now, we prove the existence of a global compact attractor for (3.55). We take an arbitrary initial condition v_0 in the ball $B^n(0, R)$ of $\left[\mathbf{R}^{2n+1}\right]^m$. For any solution $v(\cdot)$ corresponding to v_0, we have

$$\frac{1}{2}\frac{d}{dt} \|v\|^2_{\left[\mathbf{R}^{2n+1}\right]^m} + \langle a B_n v, B_n v \rangle = -\langle f_n(v), v \rangle,$$

and using (H1) and (H3), we obtain

$$\frac{1}{2}\frac{d}{dt} \|v\|^2_{\left[\mathbf{R}^{2n+1}\right]^m} + \beta \|B_n v\|^2_{\left[\mathbf{R}^{2n+1}\right]^m} + \alpha \|v\|^2_{\left[\mathbf{R}^{2n+1}\right]^m} \leq \|c_0\|_{l^1}.$$

By the Gronwall's lemma,

$$\begin{aligned} \|v(t)\|^2_{\left[\mathbf{R}^{2n+1}\right]^m} &\leq e^{-2\alpha t} \|v(0)\|^2_{\left[\mathbf{R}^{2n+1}\right]^m} + \tfrac{1}{\alpha} \|c_0\|_{l^1} \\ &\leq e^{-2\alpha t} R^2 + \tfrac{1}{\alpha} \|c_0\|_{l^1}. \end{aligned} \tag{3.56}$$

As in Sect. 3.2 from this inequality, we deduce that every local solution can be extended to a global one. Hence, we can define the multivalued semiflow $G_n(t, .) : \left[\mathbf{R}^{2n+1}\right]^m \longrightarrow P\left(\left[\mathbf{R}^{2n+1}\right]^m\right)$ as follows:

$$G_n(t, v_0) := \left\{ z \in \left[\mathbf{R}^{2n+1}\right]^m \ : \exists\, v(\cdot) \in \mathscr{D}_n(v_0) \text{ with } v(0) = v_0 \text{ and } v(t) = z \right\},$$

where $\mathscr{D}_n\left(u^0\right) = \{u(\cdot) : u \text{ is a solution of (3.55) defined for any } t \geq 0\}$. Arguing as in the proof of Lemma 3.5, we obtain that G_n is a strict multivalued semiflow.

Also, in view of (3.56) for any bounded set B, there exists $T(B)$ such that

$$\|v(t)\|^2_{\left[\mathbf{R}^{2n+1}\right]^m} \leq \frac{1}{\alpha} \|c_0\|_{l^1} + 1, \ \forall t \geq T.$$

Hence, the set

$$\theta_n := \left\{ v \in \left[\mathbf{R}^{2n+1}\right]^m : \|v\|^2_{\left[\mathbf{R}^{2n+1}\right]^m} \leq \left(\frac{1}{\alpha} \|c_0\|_{l^1} + 1 \right) \right\}$$

is absorbing for G_n. Observe that the radius of this ball does not depend on n.

Since in $\left[\mathbf{R}^{2n+1}\right]^m$ every bounded set is precompact, it follows from (3.56) that G_n is asymptotically compact.

In a similar way as in Corollaries 3.3, 3.4 and Proposition 3.3, we can prove that for all $t \geq 0$, the map $G_n(t, \cdot)$ has closed graph, compact values, and that it is upper semicontinuous.

From all these properties, using again Theorem 1.3 and Remark 1.7, we obtain the existence of a global compact attractor $\mathscr{A}_n$. Moreover, it is minimal, invariant, and $\mathscr{A}_n \subset \theta_n$, $\forall n$.

Our main goal in this section is to prove that the attractors $\mathscr{A}_n \subset \left[\mathbf{R}^{2n+1}\right]^m$, embedded in the natural way in l^2_m, converge to $\mathscr{A}$ in the following sense:

$$dist\left(\mathscr{A}_n, \mathscr{A}\right) \to 0 \text{ as } n \to \infty,$$

that is, $\mathscr{A}$ is upper semicontinuous with respect to the approximative attractors $\mathscr{A}_n$.

First, we need some uniform estimates on $\mathscr{A}_n$.

Lemma 3.9. *Assume (H1)–(H4). Then for any $\epsilon > 0$, there exists $k(\varepsilon)$ such that for $n \in \mathbf{N}$, $n \geq k(\varepsilon)$, $w \in \mathscr{A}_n$,*

$$\sum_{k(\varepsilon) \leq |i| \leq n} |w_{\cdot,i}|^2_{\mathbf{R}^m} \leq \epsilon. \tag{3.57}$$

Proof. Let $v_0 \in \mathscr{A}_n$ and take a solution $v(\cdot)$ of (3.55) such that $v(0) = v_0$. It is obvious from the invariance of the attractor that

$$v(t) \in \mathscr{A}_n \subset \theta_n, \ \forall t \in \mathbf{R}^+. \tag{3.58}$$

Take $k > 0$, to be determined later on, and $n \geq k$. Define $z(t)$ by $z_{\cdot,i}(t) := \rho_{k,i} v_{\cdot,i}(t)$. We multiply (3.55) by z to get

$$\left\langle \dot{v}, \rho_k v \right\rangle + \left\langle a B_n v, B_n \left(\rho_k v\right)\right\rangle = -\left\langle f_n\left(v\right), \rho_k v\right\rangle .$$

Arguing as in the proof of (3.42) (but using (3.58) instead of Proposition 3.2), one can obtain that

$$\left\langle a B_n v, B_n z\right\rangle \geq -\frac{C_1}{k},$$

where C_1 does not depend n.

Indeed, using (H3) for $-n \leq i \leq n-1$, we get

$$\begin{aligned}
\left(a\left(B_n v\right)_{\cdot,i}, \left(B_n z\right)_{\cdot,i}\right)_{\mathbf{R}^m} &= \rho_{k,i}\left(a\left[v_{\cdot,i+1} - v_{\cdot,i}\right], \left[v_{\cdot,i+1} - v_{\cdot,i}\right]\right)_{\mathbf{R}^m} \\
&\quad + \left(a\left[v_{\cdot,i+1} - v_{\cdot,i}\right], \left[\rho_{k,i+1} - \rho_{k,i}\right] v_{\cdot,i+1}\right)_{\mathbf{R}^m} \\
&= \rho_{k,i}\left(\frac{1}{2}\left(a + a^t\right)\left[v_{\cdot,i+1} - v_{\cdot,i}\right], \left[v_{\cdot,i+1} - v_{\cdot,i}\right]\right)_{\mathbf{R}^m} \\
&\quad + \left(a\left[v_{\cdot,i+1} - v_{\cdot,i}\right], \left[\rho_{k,i+1} - \rho_{k,i}\right] v_{\cdot,i+1}\right)_{\mathbf{R}^m} \\
&\geq \rho_{k,i}\beta\left\|v_{\cdot,i+1} - v_{\cdot,i}\right\|^2_{\mathbf{R}^m} + \left(\rho_{k,i+1} - \rho_{k,i}\right) \times \\
&\quad \times \left[\left(a v_{\cdot,i+1}, v_{\cdot,i+1}\right)_{\mathbf{R}^m} - \left(a v_{\cdot,i}, v_{\cdot,i+1}\right)_{\mathbf{R}^m}\right],
\end{aligned} \tag{3.59}$$

and

$$\begin{aligned}
\left(a\left(B_n v\right)_{\cdot,n}, \left(B_n z\right)_{\cdot,n}\right)_{\mathbf{R}^m} &= \rho_{k,n}\left(a\left[v_{\cdot,-n} - v_{\cdot,n}\right], \left[v_{\cdot,-n} - v_{\cdot,i}\right]\right)_{\mathbf{R}^m} \\
&= \rho_{k,n}\left(\tfrac{1}{2}\left(a + a^t\right)\left[v_{\cdot,-n} - v_{\cdot,n}\right], \left[v_{\cdot,-n} - v_{\cdot,i}\right]\right)_{\mathbf{R}^m} \\
&\geq \rho_{k,n}\beta\left\|v_{\cdot,-n} - v_{\cdot,n}\right\|^2_{\mathbf{R}^m} \geq 0.
\end{aligned} \tag{3.60}$$

In view of $v\left(t\right) \in A_n \subset \theta_n$, we have

$$\begin{aligned}
\sum_{i=-n}^{n-1}\left|\left(\rho_{k,i+1} - \rho_{k,i}\right)\left(a v_{\cdot,i}, v_{\cdot,i+1}\right)_{\mathbf{R}^m}\right| &\leq \sum_{i=-n}^{n-1}\sum_{j=1}^{m}\sum_{l=1}^{m}\left|\left(\rho_{k,i+1} - \rho_{k,i}\right) a_{j,l} v_{l,i} v_{j,i+1}\right| \\
&\leq \max_{1\leq j,l\leq m}\left|a_{j,l}\right| \frac{1}{2}\sum_{i=-n}^{n-1}\left|\rho_{k,i+1} - \rho_{k,i}\right| \\
&\quad \sum_{j=1}^{m}\sum_{l=1}^{m}\left(\left|v_{l,i}\right|^2 + \left|v_{j,i+1}\right|^2\right) \\
&\leq a_{\max}\frac{m}{2}\sum_{i=-n}^{n-1}\frac{\left|\theta'\left(\xi_i\right)\right|}{k}\left(\left\|v_{\cdot,i}\right\|^2_{\mathbf{R}^m} + \left\|v_{\cdot,i+1}\right\|^2_{\mathbf{R}^m}\right) \\
&\leq a_{\max}\frac{m R_1}{2k},
\end{aligned}$$

$$\sum_{i=-n}^{n-1} \left|\left(\rho_{k,i+1}-\rho_{k,i}\right)\right| \left(av_{\cdot,i+1}, v_{\cdot,i+1}\right)_{\mathbf{R}^m} \leq a_{\max} m \sum_{i=-n}^{n-1} \frac{\left|\theta'\left(\xi_i\right)\right|}{k} \|v_{\cdot,i+1}\|^2_{\mathbf{R}^m}$$
$$\leq a_{\max} \frac{mR_2}{k},$$

where $a_{\max} := \max_{1\leq j,l\leq m} \left|a_{j,l}\right|$ and R_i are constants which depends on B. Therefore, from (3.59) and (3.60), we have

$$\sum_{i=-n}^{n} \left(a\left(B_n v\right)_{\cdot,i}, \left(B_n z\right)_{\cdot,i}\right)_{\mathbf{R}^m} \geq -\frac{C_1}{k},$$

where C_1 depends on B and the parameters of the problem.

Thus, using (H1), we have

$$\frac{1}{2}\frac{d}{dt} \sum_{|i|\leq n} \rho_{k,i} \left|v_{\cdot,i}\right|^2_{\mathbf{R}^m} + \alpha \sum_{|i|\leq n} \rho_{k,i} \left|v_{\cdot,i}\right|^2_{\mathbf{R}^m} \leq \sum_{|i|\leq n} \rho_{k,i} c_{0,i} + \frac{C_1}{k}.$$

For any $\epsilon > 0$, there exists $k(\epsilon)$ such that $\sum_{|i|\leq n} \rho_{k,i} c_{0,i} + \frac{C_1}{k} \leq \frac{\epsilon\alpha}{2}$, so that

$$\frac{d}{dt} \sum_{|i|\leq n} \rho_{k,i} \left|v_{\cdot,i}\right|^2_{\mathbf{R}^m} + 2\alpha \sum_{|i|\leq n} \rho_{k,i} \left|v_{\cdot,i}\right|^2_{\mathbf{R}^m} \leq \epsilon\alpha.$$

By the Gronwall's lemma and $\mathscr{A}_n \subset \theta_n$, we have

$$\sum_{|i|\leq n} \rho_{k,i} \left|v_{\cdot,i}(t)\right|^2_{\mathbf{R}^m} \leq \frac{\epsilon}{2} + e^{-2\alpha t} \|v(0)\|^2_{\left[\mathbf{R}^{2n+1}\right]^m}$$
$$\leq \frac{\epsilon}{2} + e^{-2\alpha t} R_0^2,$$

where $R_0^2 := 1 + \frac{\|c_0\|_{l^1}}{\alpha}$. Taking $T_1 = \frac{1}{2\alpha}\log\left(\frac{2R_0^2}{\epsilon}\right)$, one has $e^{-\alpha t} R_0^2 \leq \frac{\epsilon}{2}$, if $t \geq T_1$, and then,

$$\sum_{2k(\epsilon)\leq|i|\leq n} \left|v_{\cdot,i}(t)\right|^2_{\mathbf{R}^m} \leq \sum_{|i|\leq n} \rho_{k,i} \left|v_{\cdot,i}(t)\right|^2_{\mathbf{R}^m} \leq \epsilon, \ \forall t \geq T_1. \tag{3.61}$$

Finally, let $w \in \mathscr{A}_n$ be arbitrary. Since $G_n\left(T_1, \mathscr{A}_n\right) = \mathscr{A}_n$, we can state the existence of $v_0 \in \mathscr{A}_n$ such that $w \in G\left(T_1, v_0\right)$, and then, we get a solution $v(\cdot)$ satisfying $v\left(T_1\right) = w$ and $v(0) = v_0$. From (3.61), we have

$$\sum_{2k(\epsilon)\leq|i|\leq n} \left|w_{\cdot,i}\right|^2_{\mathbf{R}^m} = \sum_{2k(\epsilon)\leq|i|\leq n} \left|v_{\cdot,i}\left(T_1\right)\right|^2_{\mathbf{R}^m} \leq \epsilon.$$

□

Remark 3.4. For $w \in \left[\mathbf{R}^{2n+1}\right]^m$, we shall use the embedding in l_m^2 defined by

$$\hat{w} := \left\{ \begin{array}{c} w_{\cdot,i} \in \mathbf{R}^m, \text{ if } |i| \leq n \\ 0, \text{ if } |i| > n \end{array} \right\}.$$

We identify w and $\hat{w}$ when no confusion is possible. Hence, $\left[\mathbf{R}^{2n+1}\right]^m \ni v_n \to v_0$ in l_m^2 will mean that $\hat{v}_n \to v_0$ in l_m^2.

Now, we are ready to prove the upper semicontinuity of the attractor.

Lemma 3.10. *Assume (H1)–(H4). Then dist* $(\mathscr{A}_n, \mathscr{A}) \to 0$, *if* $n \to \infty$.

Proof. First, we check that $K := \overline{\cup_{n=1}^{\infty} \mathscr{A}_n}$ is compact in l_m^2. It is sufficient to prove that an arbitrary sequence $v^k \in B = \cup_{n=1}^{\infty} \mathscr{A}_n$ is precompact. As $\mathscr{A}_n \subset \theta_n$, the set B is bounded. Therefore, up to a subsequence, $v^k \to v$ weakly in l_m^2. We have to see that this convergence is strong. By (3.57), we know that, given $\epsilon > 0$ arbitrary, there exists $K(\epsilon)$ such that $\sum_{|i|>K(\epsilon)} |w_{\cdot,i}|^2_{\mathbf{R}^m} \leq \epsilon$, for all $w \in B$, where $K(\epsilon)$ does not depend on n. Hence, $\sum_{|i|>K(\epsilon)} \left|v^k_{\cdot,i}\right|^2_{\mathbf{R}^m} \leq \epsilon$. Obviously, we can choose $K(\epsilon)$ satisfying also $\sum_{|i|>K(\epsilon)} |v_{\cdot,i}|^2_{\mathbf{R}^m} \leq \epsilon$.

Then there exists $N(\epsilon, K)$ such that for $k \geq N$,

$$\begin{aligned} \left\|v^k - v\right\|^2_{l_m^2} &= \sum_{|i|\leq K(\epsilon)} \left|v^k_{\cdot,i} - v_{\cdot,i}\right|^2_{\mathbf{R}^m} + \sum_{|i|>K(\epsilon)} \left|v^k_{\cdot,i} - v_{\cdot,i}\right|^2_{\mathbf{R}^m} \\ &\leq \epsilon + 2\sum_{|i|>K(\epsilon)} |v_{\cdot,i}|^2_{\mathbf{R}^m} + 2\sum_{|i|>K(\epsilon)} \left|v^k_{\cdot,i}\right|^2_{\mathbf{R}^m} \leq 5\epsilon, \end{aligned}$$

proving that $v^k \to v$ strongly in l_m^2.

Further, in a similar way as in the proof of (3.47) (but using now (3.57)), one can prove that for a fixed $T > 0$ if $v^0 \in l_m^2$, $v^{0,n} \in \mathscr{A}_n$ are such that $v^{0,n} \to v^0$ in l_m^2, then for any sequence of solutions $v^n(\cdot) \in \mathscr{D}_n\left(v^{0,n}\right)$ verifying $v^n(0) = v^{0,n}$, there exists a subsequence such that

$$v^{n_k}(\cdot) \to v(\cdot) \in \mathscr{D}\left(v^0\right) \text{ in } C\left([0,T], l_m^2\right). \tag{3.62}$$

Indeed, fix $t \in [0,T]$. In view of $v^n(t) \in \mathscr{A}_n \subset \theta_n$, passing to a subsequence, we can state that $v^n(t) \to w$ weakly in l_m^2. Then by (4.56) for any $\sigma > 0$, there exist $\widetilde{N}(\sigma)$ and $\widetilde{K}(\sigma)$ such that

$$\begin{aligned} \|v^n(t) - w\|_{l_m^2} &\leq \sum_{|i|\leq \widetilde{K}(\sigma)} \left|v^n_{\cdot,i}(t) - w_{\cdot,i}\right|^2_{\mathbf{R}^m} + \sum_{\widetilde{K}(\sigma)<|i|\leq n} \left|v^n_{\cdot,i}(t) - w_{\cdot,i}\right|^2_{\mathbf{R}^m} \\ &\leq \sum_{|i|\leq \widetilde{K}(\sigma)} \left|v^n_{\cdot,i}(t) - w_{\cdot,i}\right|^2_{\mathbf{R}^m} + 2\sum_{\widetilde{K}(\sigma)<|i|\leq n} \left|v^n_{\cdot,i}(t)\right|^2_{\mathbf{R}^m} \\ &\quad + 2\sum_{\widetilde{K}(\sigma)<|i|\leq n} |w_{\cdot,i}(t)|^2_{\mathbf{R}^m} < \sigma, \end{aligned}$$

if $n \geq \widetilde{N}$. Hence, $v^n(t) \to w$ strongly in l_m^2. Therefore, the sequence $v^n(t)$ is precompact for any t.

Further, arguing as in the case of the map F, one can obtain that

$$\|aA_n v^n(t) + f_n(v^n(t))\|_{l_m^2} \leq C, \ \forall n \in \mathbf{N} \text{ and } t \in [0, T],$$

and from the equality $\frac{d}{dt}v^n = -aA_n v^n(t) - f_n(v^n(t))$, we obtain that

$$\left\| \frac{d}{dt} v^n(t) \right\|_{l_m^2} \leq C,$$

proving that the sequence v^n is equicontinuous. The Ascoli–Arzelà theorem implies then the existence of a subsequence v^{n_k} converging in $C\left([0,T], l_m^2\right)$ to some function $v(\cdot)$.

We shall prove that $A_n v^n(t) \to Av(t)$, $f_n(v^n(t)) \to \widehat{f}(v(t))$ in l_m^2, uniformly in $t \in [0,T]$, where for $A_n v^n(t)$, $f_n(v^n(t))$, we are considering the standard embedding in l_m^2 given in Remark 3.4. On the one hand, by (H2), we have

$$\begin{aligned}
\left\| f_n(v^n(t)) - \widehat{f}(v(t)) \right\|_{l_m^2}^2 &\leq 2 \left\| f_n(v^n(t)) - \widehat{f}(v^n(t)) \right\|_{l_m^2}^2 \\
&\quad + 2 \left\| \widehat{f}(v^n(t)) - \widehat{f}(v(t)) \right\|_{l_m^2}^2 \\
&= 2 \sum\nolimits_{|i|>n} |f_{\cdot,i}(0)|_{\mathbf{R}^m}^2 + 2 \left\| \widehat{f}(v^n(t)) - \widehat{f}(v(t)) \right\|_{l_m^2}^2 \\
&\leq 2 \sum\nolimits_{|i|>n} c_{1,i}^2 + 2 \left\| \widehat{f}(v^n(t)) - \widehat{f}(v(t)) \right\|_{l_m^2}^2 .
\end{aligned}$$

As $\widehat{f}(v^n(t)) \to \widehat{f}(v(t))$ uniformly in $[0,T]$, we obtain $f_n(v^n(t)) \to \widehat{f}(v(t))$ in l_m^2 uniformly in $t \in [0,T]$.

On the other hand, we note that for all j,

$$\begin{aligned}
\left(\widehat{A}_n v_{j,\cdot}^n(t) - \widehat{A} v_{j,\cdot}^n(t) \right)_i &= 0 \text{ if } |i| \leq n-1 \text{ or } |i| \geq n+2, \\
\left(\widehat{A}_n v_{j,\cdot}^n(t) - \widehat{A} v_{j,\cdot}^n(t) \right)_{-n} &= -v_{j,n}^n, \\
\left(\widehat{A}_n v_{j,\cdot}^n(t) - \widehat{A} v_{j,\cdot}^n(t) \right)_n &= -v_{j,-n}^n, \\
\left(\widehat{A}_n v_{j,\cdot}^n(t) - \widehat{A} v_{j,\cdot}^n(t) \right)_{-n-1} &= v_{j,-n}^n, \\
\left(\widehat{A}_n v_{j,\cdot}^n(t) - \widehat{A} v_{j,\cdot}^n(t) \right)_{n+1} &= v_{j,n}^n.
\end{aligned}$$

Then

$$\|A_n v^n(t) - Av(t)\|^2_{l^2_m} \leq 2\|A_n v^n(t) - Av^n(t)\|^2_{l^2_m} + 2\|Av^n(t) - Av(t)\|^2_{l^2_m}$$
$$= 4\left(\left|v^n_{\cdot,n}(t)\right|^2_{\mathbf{R}^m} + \left|v^n_{\cdot,-n}(t)\right|^2_{\mathbf{R}^m}\right)$$
$$+ 2\|Av^n(t) - Av(t)\|^2_{l^2_m} \to 0,$$

uniformly in $t \in [0, T]$, in view of $Av^n(t) - Av(t) \to 0$, uniformly in $t \in [0, T]$, and (3.57).

Thus, we obtain using the integral representation of solutions that v is a solution. Also, it is clear that $v(0) = v^0$.

Finally, we note that

$$dist(\mathscr{A}_n, \mathscr{A}) \leq dist(G_n(t, \mathscr{A}_n), \mathscr{A})$$
$$\leq dist(G_n(t, \mathscr{A}_n), G(t, K)) + dist(G(t, K), \mathscr{A}).$$

Since $\mathscr{A}$ attracts any bounded set in l^2_m, for any $\epsilon > 0$, we can find t_0 satisfying

$$dist(G(t, K), \mathscr{A}) \leq \frac{\epsilon}{2}, \forall t \geq t_0.$$

Fix $t \geq t_0$. Let us prove the existence of $N(\epsilon)$ such that $dist(G_n(t, \mathscr{A}_n), G(t, K)) \leq \frac{\epsilon}{2}$, if $n \geq N(\epsilon)$. By contradiction, suppose that for some $\epsilon > 0$, there exists a sequence n_k such that $dist\left(G_{n_k}\left(t, \mathscr{A}_{n_k}\right), G(t, K)\right) > \frac{\epsilon}{2}$. Then we can find $v_{n_k} \in G_{n_k}\left(t, \mathscr{A}_{n_k}\right)$ satisfying

$$dist\left(v_{n_k}, G(t, K)\right) > \frac{\epsilon}{2}. \tag{3.63}$$

Let $v_{0,n_k} \in \mathscr{A}_{n_k}$, $v_{n_k}(\cdot) \in \mathscr{D}_{n_k}\left(v_{0,n_k}\right)$ be such that $v_{n_k}(0) = v_{0,n_k}$ and $v_{n_k}(t) = v_{n_k}$. Since K is compact, passing to a subsequence, there exists $v_0 \in K$ such that $v_{0,n_k} \to v_0$ in l^2_m. Now, using (3.62), we find $v(\cdot)$ for which $v_{n_k}(\cdot) \to v(\cdot)$ in $C\left([0, t]; l^2_m\right)$. Hence, $v_{n_k}(t) = v_{n_k} \to v(t) = v$ in l^2_m. Clearly, $v(0) = v_0$, so that $v(t) = v \in G(t, K)$. Therefore, there exists $N(\epsilon)$ satisfying $\|v_{n_k} - v\|_{l^2_m} \leq \frac{\epsilon}{2}$, if $n_k \geq N$, that is, $dist\left(v_{n_k}, G(t, K)\right) \leq \frac{\epsilon}{2}$, which is a contradiction with (3.63). Hence,

$$dist(\mathscr{A}_n, \mathscr{A}) \leq \epsilon, \text{ if } n \geq N(\epsilon).$$

□

3.5 Application for Discrete Climate Energy Balance Model

We now consider a climate energy balance model (see Example 4). The problem is the following:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + Bu = QS(x)\beta(u) + h(x), (t, x) \in \mathbf{R}_+ \times \mathbf{R},$$
$$u(x, 0) = u_0(x), x \in \mathbf{R}, \tag{3.64}$$

where B and Q are positive constants, S, $h \in L_\infty(\mathbf{R})$, $u_0 \in L_2(\mathbf{R})$, and β is univocal and has maximal monotone graph in $\mathbf{R}^2$, which is bounded, that is, there exist $m, M \in \mathbf{R}$ such that

$$m \le z \le M, \quad \text{for all } z \in \beta(s), s \in \mathbf{R}. \tag{3.65}$$

We also assume that

$$0 < S_0 \le S(x) \le S_1, \quad \text{a.e. } x \in \mathbf{R}. \tag{3.66}$$

The unknown $u(t, x)$ represents the averaged temperature of the Earth surface, Q is the so-called solar constant, which is the average (over a year and over the surface of the Earth) value of the incoming solar radiative flux, and the function $S(x)$ is the insolation function given by the distribution of incident solar radiation at the top of the atmosphere. When the averaging time is of the order of 1 year or longer, the function $S(x)$ satisfies (3.66); for shorter periods, we must assume that $S_0 = 0$. The term β represents the so-called co-albedo function. It represents the ratio between the absorbed solar energy and the incident solar energy at the point x on the Earth surface (see Example 4). Let us set in (3.1): $a = 1$, $f(x, u) = Bu - QS(x)\beta(u) - h(x)$ for a.e. x. After averaging and discretization, we get the appropriate discrete model (3.2) for which all results from this chapter are fulfilled.

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Chapter 4
On Global Attractors of Multivalued Semiprocesses and Nonautonomous Evolution Inclusions

In the first chapter, we considered the existence and properties of global attractors for autonomous multivalued dynamical systems.

When the equation is nonautonomous, new and challenging difficulties appear. In this case, if uniqueness of the Cauchy problem holds, then the usual semigroup of operators becomes a two-parameter semigroup or process [38, 39], as we have to take into account the initial and the final time of the solutions. An alternative method is to transform the system into an autonomous one by using the skew product formalism. We note that in real applications systems are usually nonautonomous, as it is natural to expect that the parameters of the problem depend on time.

Nevertheless, when the nonautonomous terms are periodic or almost periodic, the asymptotic behavior of solutions (in particular the existence of uniform global attractors) can be studied in a similar way as in the autonomous case. In [10], a nice theory was constructed for abstract processes and semiprocesses with applications to nonautonomous reaction-diffusion equations, Navier–Stokes equations and hyperbolic equations. This theory is based in the autonomous definition of global attractor and it has been applied in several different models (see, e.g., [3, 4, 9, 16–19, 21, 27–29, 32, 35, 36, 43, 45, 46]).

As in the autonomous case when uniqueness of the Cauchy problem fails, the theory of processes is no more applicable. In this chapter, we generalize the construction of [10, 12] to multivalued semiprocesses and processes giving an abstract theory which can be applied to a large class of nonautonomous equations. First, we define multivalued semiprocesses and study their ω-limit sets and global attractors. Then the abstract theorems are applied to evolution inclusions of the type

$$\begin{cases} \frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) \in f\,(t,u) + g\,(t)\,, \text{ in } \Omega \times (\tau, T)\,, \\ u\mid_{\partial\Omega} = 0,\, u\,(\tau) = u_0, \end{cases}$$

where $\tau \geq 0$, $\Omega \subset \mathbf{R}^n$ is a bounded open subset, $p \geq 2$, $g \in L_\infty\,(0, +\infty; L_2\,(\Omega))$ and $f : \mathbf{R}_+ \times \mathbf{R} \to 2^{\mathbf{R}}$ is a Lipschitz (in the multivalued sense) multivalued map with nonempty, convex, compact values. Under some additional conditions, it is proved

M.Z. Zgurovsky et al., *Evolution Inclusions and Variation Inequalities for Earth Data Processing III*, Advances in Mechanics and Mathematics 27,
DOI 10.1007/978-3-642-28512-7_4, © Springer-Verlag Berlin Heidelberg 2012

that this inclusion generates a multivalued semiprocess having a uniform compact global attractor.

These results were mainly proved in [34]. In [20, 22–24, 30], this theory was generalized for multivalued semiprocesses and processes in topological spaces and applied to reaction-diffusion, phase-field equations and the three-dimensional Navier–Stokes system. Kernel sections for multivalued semiprocesses have been studied in [44].

We observe that when the nonlinear term is more general, the concept of uniform attractor is not useful. In such a case, the notion of pullback attractor became the appropriate concept. The definition of pullback attractor appeared meanly motivated by the application to stochastic differential equations [13] (see [7] for the extension to the multivalued case) in the framework of cocycle maps, but it became soon useful for general nonautonomous equations (see, e.g., [6, 8, 25, 37] and the references therein). We will write more detaily about this concept in Chap. 5.

4.1 ω-Limit Sets and Global Attractors of Multivalued Semiprocesses

Let X be a complete metric space with the metric ρ; 2^X $(P(X), \beta(X), C(X), K(X))$ be the set of all (nonempty, nonempty bounded, nonempty closed, nonempty compact) subsets of the space X, and let Σ be a compact metric space. Denote $\mathbf{R}_+ = [0, +\infty)$, $\mathbf{R}_+^2 = \mathbf{R}_+ \times \mathbf{R}_+$, $\mathbf{R}_{+d} = \left\{(t, \tau) \in \mathbf{R}_+^2 : t \geq \tau\right\}$. For $A, B \in \beta(X)$ and $x \in X$ set $\operatorname{dist}(x, B) = \inf_{y \in B} \rho(x, y)$, $\operatorname{dist}(A, B) = \sup_{x \in A} \operatorname{dist}(x, B)$, $O_\delta(A) = \{y \in X : \operatorname{dist}(y, A) < \delta\}$, where $\delta > 0$.

Definition 4.1. The map $U : \mathbf{R}_{+d} \times X \to P(X)$ is called a *multivalued semiprocess (MSP)* if:

1. $U(t, t, \cdot) = Id$ is the identity map.
2. $U(t, \tau, x) \subset U(t, s, U(s, \tau, x))$, $\forall t \geq s \geq \tau, \forall x \in X$, where $U(t, s, U(s, \tau, x)) = \bigcup_{y \in U(s,\tau,x)} U(t, s, y)$.

The multivalued semiprocess U is called *strict* if, moreover,

$$U(t, s, U(s, \tau, x)) = U(t, \tau, x), \forall t \geq s \geq \tau, \forall x \in X.$$

Consider the family of MSP $\{U_\sigma : \sigma \in \Sigma\}$ and define the map $U_+ : \mathbf{R}_{+d} \times X \to P(X)$ by

$$U_+(t, \tau, x) = \bigcup_{\sigma \in \Sigma} U_\sigma(t, \tau, x).$$

Remark 4.1. It is clear that U_+ is an MSP.

Definition 4.2. The set $A \subset X$ is called uniformly attracting for the family of MSP U_σ if for any $B \in \beta(X)$ and $\tau \in \mathbf{R}_+$,

$$\lim_{t \to +\infty} \operatorname{dist}(U_+(t, \tau, B), A) = 0. \tag{4.1}$$

For $B \subset X$, define $\gamma_{s,\sigma}^{\tau}(B) = \cup_{t \geq s} U_{\sigma}(t, \tau, B)$ and

$$\omega_{\tau,\Sigma}(B) = \bigcap_{t \geq \tau} \overline{\bigcup_{\sigma \in \Sigma} \gamma_{t,\sigma}^{\tau}(B)}.$$

Definition 4.3. The family of MSP $\{U_{\sigma} : \sigma \in \Sigma\}$ is called uniformly asymptotically upper semicompact if for any $B \in \beta(X)$ and $\tau \in \mathbf{R}_{+}$ such that for some $T = T(B, \tau)$, $\gamma_{T,\Sigma}^{\tau}(B) = \bigcup_{\sigma \in \Sigma} \gamma_{T,\sigma}^{\tau}(B) \in \beta(X)$, any sequence $\{\xi_n\}$, $\xi_n \in U_{\sigma_n}(t_n, \tau, B)$, $\sigma_n \in \Sigma$, $t_n \to +\infty$, is precompact in X.

Proposition 4.1. *Let the family of MSP $\{U_{\sigma} : \sigma \in \Sigma\}$ be uniformly asymptotically upper semicompact, and for any $B \in \beta(X)$ and $\tau \in \mathbf{R}_{+}$, there exist $T = T(B, \tau)$ such that $\gamma_{T,\Sigma}^{\tau}(B) = \bigcup_{\sigma \in \Sigma} \gamma_{T,\sigma}^{\tau}(B) \in \beta(X)$.*

Then $\omega_{\tau,\Sigma}(B) \neq \emptyset$, $\forall B \in \beta(X)$, $\tau \in \mathbf{R}_{+}$. Moreover, it is compact in X and

$$\operatorname{dist}(U_{+}(t, \tau, B), \omega_{\tau,\Sigma}(B)) \to 0, \text{ as } t \to +\infty.$$

If P is another closed set such that $\operatorname{dist}(U_{+}(t, \tau, B), P) \underset{t \to +\infty}{\to} 0$, then $\omega_{\tau,\Sigma}(B) \subset P$ (minimality property).

Proof. The set $\omega_{\tau,\Sigma}(B)$ can be characterized as follows.

Lemma 4.1. *The following statements are equivalent:*

1. *$y \in \omega_{\tau,\Sigma}(B)$.*
2. *There exists a sequence $\{\xi_n\}$ such that $\xi_n \in U_{\sigma_n}(t_n, \tau, B)$ and $\xi_n \to y \in X$, where $t_n \to +\infty$ and $\sigma_n \in \Sigma$.*

Proof. Consider the sequence of sets $\gamma_{t_n,\Sigma}^{\tau}(B)$, where $t_n \to +\infty$ as $n \to +\infty$. The upper and lower limits of these sets are defined by

$$\operatorname{Lim\,sup}\left(\gamma_{t_n,\Sigma}^{\tau}(B)\right) = \left\{ y \in X : \liminf_{n \to +\infty} \operatorname{dist}\left(y, \gamma_{t_n,\Sigma}^{\tau}(B)\right) = 0 \right\},$$

$$\operatorname{Lim\,inf}\left(\gamma_{t_n,\Sigma}^{\tau}(B)\right) = \left\{ y \in X : \lim_{n \to +\infty} \operatorname{dist}\left(y, \gamma_{t_n,\Sigma}^{\tau}(B)\right) = 0 \right\}.$$

It follows from these definitions that $y \in \operatorname{Lim\,inf}\left(\gamma_{t_n,\Sigma}^{\tau}(B)\right)$ ($\operatorname{Lim\,sup}\left(\gamma_{t_n,\Sigma}^{\tau}(B)\right)$, respectively) if and only if for any neighborhood $O(y)$ of y, there exists n_0 such that $O(y) \cap \gamma_{t_n,\Sigma}^{\tau}(B) \neq \emptyset$ for $n \geq n_0$ (respectively for $n_k \geq n_0$, where n_k is some subsequence). Since $\gamma_{t_2,\Sigma}^{\tau}(B) \subset \gamma_{t_1,\Sigma}^{\tau}(B)$ if $t_2 \geq t_1$,

$$\begin{aligned} \operatorname{Lim\,sup}\left(\gamma_{t_n,\Sigma}^{\tau}(B)\right) = \operatorname{Lim\,inf}\left(\gamma_{t_n,\Sigma}^{\tau}(B)\right) &= \bigcap_{t_n \geq \tau} \overline{\gamma_{t_n,\Sigma}^{\tau}(B)} \\ &= \omega_{\tau,\Sigma}(B) \end{aligned} \tag{4.2}$$

(see [2, p.18]).

Let $y \in \omega_{\tau,\Sigma}(B)$. Then in view of (4.2) for any neighborhood $O_\delta(y)$, $\delta > 0$, of y, there exists n_0 such that $O_\delta(y) \cap \gamma^\tau_{t_n,\Sigma}(B) \neq \emptyset$ for $n \geq n_0$. Let us take a sequence $\delta_n \to 0$, as $n \to +\infty$, and $\xi_n \in O_{\delta_n}(y) \cap \gamma^\tau_{t_n,\Sigma}(B)$. Hence, $\xi_n \in U_{\sigma_n}(\tau_n, \tau, B)$, where $\tau_n \geq t_n$, and $\xi_n \to y$, as $n \to +\infty$.

Let now $\xi_n \in U_{\sigma_n}(t_n, \tau, B)$ be such that $\xi_n \to y \in X$, where $t_n \to +\infty$ and $\sigma_n \in \Sigma$. Then $\xi_n \in \gamma^\tau_{t_n,\Sigma}(B)$, and for any neighborhood $O(y)$, there exists n_0 for which $\xi_n \in O(y)$ if $n \geq n_0$. Hence, $O(y) \cap \gamma^\tau_{t_n,\Sigma}(B) \neq \emptyset$, $\forall n \geq n_0$, so that $y \in \omega_{\tau,\Sigma}(B)$. □

Further, let us prove that $\omega_{\tau,\Sigma}(B) \neq \emptyset$. If this is not the case, then Lemma 4.1 implies that there cannot be converging sequences $\xi_n \in U_{\sigma_n}(t_n, \tau, B)$, where $t_n \to +\infty$. But as the family of MSP $\{U_\sigma : \sigma \in \Sigma\}$ is uniformly asymptotically upper semicompact, we can extract a converging subsequence, which is a contradiction.

Let $\{\xi_n\} \subset \omega_{\tau,\Sigma}(B)$ be an arbitrary sequence. It follows that $\xi_n \in \overline{\gamma^\tau_{t,\Sigma}(B)}$, $\forall t \geq \tau$. Therefore, there exist sequences $\{t_n\}, \{\zeta_n\}$ such that $\zeta_n \in U_{\sigma_n}(t_n, \tau, B)$, $\sigma_n \in \Sigma$, $t_n \to +\infty$ as $n \to +\infty$, and $\rho(\zeta_n, \xi_n) < \frac{1}{n}$, $\forall n$. Using again Lemma 4.1 and the fact that U_σ is uniformly upper semicompact, we obtain the existence of a converging subsequence $\zeta_{n_k} \to \zeta_0 \in \omega_{\tau,\Sigma}(B)$. But then, $\xi_{n_k} \to \zeta_0$. Hence, $\omega_{\tau,\Sigma}(B)$ is compact.

Suppose now that there exist $\varepsilon > 0$ and $y_n \in U_{\sigma_n}(t_n, \tau, B)$, $\sigma_n \in \Sigma$, $t_n \to +\infty$, such that

$$\operatorname{dist}(y_n, \omega_{\tau,\Sigma}(B)) > \varepsilon, \ \forall n.$$

Then as before, we can extract a subsequence $y_{n_k} \to \zeta \in \omega_{\tau,\Sigma}(B)$, which is a contradiction. Therefore $\operatorname{dist}(U_+(t,\tau,B), \omega_{\tau,\Sigma}(B)) \to 0$, as $t \to +\infty$.

Finally, let $P \subset X$ be a closed set such that $\operatorname{dist}(U_+(t,\tau,B), P) \to 0$, as $t \to +\infty$. Lemma 4.1 implies that for any $y \in \omega_{\tau,\Sigma}(B)$, there exists $\xi_n \in U_{\sigma_n}(t_n, \tau, B)$ converging to y as $t_n \to +\infty$. Hence, $y \in P$. □

Proposition 4.2. *Let for any $B \in \beta(X)$ and $\tau \in \mathbf{R}_+$, there exist a compact set $K(B,\tau)$ such that*

$$\operatorname{dist}(U_+(t,\tau,B), K(B,\tau)) \to 0, \text{ as } t \to +\infty.$$

Then the conditions of Proposition 4.1 hold.

Proof. It is evident that for any $B \in \beta(X)$ and $\tau \in \mathbf{R}_+$ there exists $T = T(B,\tau)$ such that $\gamma^\tau_{T,\Sigma}(B) \in \beta(X)$. Consider an arbitrary sequence $\xi_n \in U_{\sigma_n}(t_n, \tau, B)$, $\sigma_n \in \Sigma$, $t_n \to +\infty$. Take a sequence $\delta_k \to 0$, as $k \to +\infty$. Then there exist a subsequence $\{t_{n_k}\}$ and $\zeta_{n_k} \in K(B,\tau)$ satisfying $\rho\left(\xi_{n_k}, \zeta_{n_k}\right) \leq \delta_k$. Since $K(B,\tau)$ is compact, we can assume (taking a subsequence if necessary) that $\zeta_{n_k} \to \zeta$. Hence, $\xi_{n_k} \to \zeta$ and U_σ is uniformly upper semicompact. □

Proposition 4.3. *Let X be an infinite-dimensional Banach space and the conditions of Proposition 4.1 (or 4.2) hold. Then for any $\tau \in \mathbf{R}_+$, there exists a set $\Theta_\tau \neq X$ such that $\forall B \in \beta(X)$*

$$\operatorname{dist}\left(U_{+}\left(t,\tau,B\right),\Theta_{\tau}\right)\to 0,\ as\ t\to+\infty. \tag{4.3}$$

Moreover, for any closed set Y_τ satisfying (4.3), $\Theta_\tau \subset Y_\tau$.

Proof. Let $\{B_i\}$ be a countable set of balls in X centered at 0 with radius i, and set $\Theta_\tau = \bigcup_{i=1}^{\infty} \omega_{\tau,\Sigma}(B_i)$. We note that if B^1 and B^2, then $\omega_{\tau,\Sigma}\left(B^1\right) \subset \omega_{\tau,\Sigma}\left(B^2\right)$ and also that any $B \in \beta(X)$ belongs to some B_k. Hence, Proposition 4.1 implies that (4.3) holds. It follows also from Proposition 4.1 that if Y_τ is a closed set satisfying (4.3), then $\omega_{\tau,\Sigma}(B) \subset Y_\tau$, $\forall B \in \beta(X)$, so that $\Theta_\tau \subset Y_\tau$. Finally, since a compact set in an infinite-dimensional Banach space is of first category and Θ_τ is a countable union of compact sets, the inequality $\Theta_\tau \neq X$ follows from Baire's theorem. □

Definition 4.4. The set Θ_Σ is a uniform global attractor for the family of MSP $\{U_\sigma : \sigma \in \Sigma\}$ if:

1. Θ_Σ is a uniformly attracting set.
2. $\Theta_\Sigma \subset U_+(t, 0, \Theta_\Sigma)$, $\forall t \in \mathbf{R}_+$.
3. For any closed uniformly attracting set Y, $\Theta_\Sigma \subset Y$.

Let Z be a topological space and $F(\mathbf{R}_+, Z)$ be some space of functions with values in Z. We shall further assume that:

(L1) $\Sigma \subset F(\mathbf{R}_+, Z)$ is a compact metric space.
(L2) On Σ is defined the continuous shift operator $T(h)\sigma(t) = \sigma(t+h)$, $h \in \mathbf{R}_+$, and $T(h)\Sigma \subseteq \Sigma$.
(L3) For any $(t,\tau) \in \mathbf{R}_{+d}$, $\sigma \in \Sigma$, $h \in \mathbf{R}_+$, $x \in X$, we have

$$U_\sigma(t+h, \tau+h, x) \subset U_{T(h)\sigma}(t,\tau,x).$$

Theorem 4.1. *Let X be an infinite-dimensional Banach space, the conditions of Proposition 4.1 be satisfied, and $L1 - L3$ hold. Suppose that for $t \geq 0$, the map $\Sigma \times X \ni (\sigma, x) \mapsto U_\sigma(t, 0, x)$ has closed graph. Then the set*

$$\Theta_\Sigma \overset{def}{=} \bigcup_{B\in\beta(X)} \omega_{0,\Sigma}(B) = \bigcup_{\tau\geq 0} \bigcup_{B\in\beta(X)} \omega_{\tau,\Sigma}(B) \tag{4.4}$$

is a uniform global attractor and $\Theta_\Sigma \neq X$. It is σ-compact and Lindelöf in X. It is locally compact in the sum topology $\tau_\oplus$.

Remark 4.2. If X is not infinite dimensional or is a metric space, all the statements remain true except the fact that $\Theta_\Sigma \neq X$.

Proof. Define the multivalued map $G : \mathbf{R}_+ \times X \times \Sigma \to P(X \times \Sigma)$ by

$$G(t,(x,\sigma)) = \left(U_\sigma(t,0,x), T(t)\sigma\right). \tag{4.5}$$

Lemma 4.2. *The map G is a multivalued semiflow in the sense of Definition 1.1, and for any $t \geq 0$, the graph of $G(t,\cdot)$ is closed.*

Proof. First, we have to prove that G is a multivalued semiflow, that is, $G(0,\cdot) = Id_{X\times\Sigma}$ (the identity map) and $G(t_1+t_2) \subset G(t_1, G(t_2,\xi))$, $\forall \xi \in X\times\Sigma$, $\forall t_1, t_2 \in \mathbf{R}_+$. Since $T(\cdot) : \Sigma \to \Sigma$ is a semigroup, $G(0,(x,\sigma)) = (x,\sigma)$. For the second property, note that in view of Definition 4.1 and $L3$

$$\begin{aligned} G(t_1+t_2,(x,\sigma)) &= (U_\sigma(t_1+t_2,0,x), T(t_1+t_2)\sigma) \\ &\times \subset (U_\sigma(t_1+t_2,t_2,U_\sigma(t_2,0,x)), T(t_1)T(t_2)\sigma) \\ &\times \subset \left(U_{T(t_2)\sigma}(t_1,0,U_\sigma(t_2,0,x)), T(t_1)T(t_2)\sigma\right) \\ &= G(t_1,(U_\sigma(t_2,0,x), T(t_2)\sigma)) = G(t_1, G(t_2,(x,\sigma))). \end{aligned}$$

Finally, the fact that $G(t,\cdot)$ has closed graph for $t \geq 0$ is a consequence of being the product of the maps $(x,\sigma) \longmapsto U_\sigma(t,0,x)$, $T(t) : \Sigma \to \Sigma$, which have closed graph. □

For $C \in \beta(X\times\Sigma)$ set $\gamma_t^+(C) = \underset{s\geq t}{\cup} G(s,C)$, $\omega(C) = \underset{t\geq 0}{\cap} \overline{\gamma_t^+(C)}$.

Lemma 4.3. *The multivalued semiflow G is asymptotically upper semicompact, i.e. $\forall C \in \beta(X\times\Sigma)$ such that $\gamma_{t_1}^+(C) \in \beta(X\times\Sigma)$ for some $t_1(C)$ any sequence $\xi_n \in G(t_n,C)$, where $t_n \to +\infty$, is precompact in $X\times\Sigma$.*

Proof. Consider an arbitrary sequence $\{\xi_n\}$ of the type described in the statement. Then $\xi_n = (y_n,\beta_n)$, where $y_n \in U_{\sigma_n}(t_n,\tau,B)$, $\beta_n = T(t_n)\sigma_n$, $B \in \beta(X)$, $\sigma_n \in \Sigma$. The sequence β_n is precompact in Σ because the space Σ is compact and $\{y_n\}$ is precompact because the family U_σ is asymptotically upper semicompact. Hence, ξ_n is precompact in $X\times\Sigma$. □

Lemma 4.4. *For any $C \in \beta(X\times\Sigma)$, there exists $T(C)$ such that $\gamma_T^+(C) \in \beta(X\times\Sigma)$.*

Proof. First, we note that $C \subset B\times\Sigma$ for some $B \in \beta(X)$. There exists $T(B)$ for which $\gamma_{T,\Sigma}^0(B) \in \beta(X)$. Hence,

$$\gamma_T^+(C) \subset \underset{t\geq T}{\cup} (U_+(t,0,B),\Sigma) = \left(\gamma_{T,\Sigma}^0(B),\Sigma\right) \in \beta(X\times\Sigma).$$

□

The previous lemmas, Theorem 1.1, and Remark 1.2 imply that for any $C \in \beta(X\times\Sigma)$, the ω-limit set $\omega(C)$ is nonempty, compact, and negatively invariant. Moreover, it attracts C,

$$\mathrm{dist}_{X\times\Sigma}(G(t,C),\omega(C)) \to 0, \text{ as } t\to\infty,$$

and it is the minimal closed set with this property, that is, if P is closed and attracts C, then $\omega(C) \subset P$.

Further, Theorem 1.1 and Remark 1.6 imply that G has the global attractor $\Re$ satisfying:

1. $\Re = \bigcup_{C\in\beta(X\times\Sigma)} \omega(C)$ and $\Re \neq X\times\Sigma$.
2. $\Re \subset G(t,\Re)$, $\forall t\in\mathbf{R}_+$.
3. $\forall C \in \beta(X\times\Sigma)$, $\mathrm{dist}_{X\times\Sigma}(G(t,C),\Re) \to 0$, as $t\to+\infty$ ($\Re$ attracts every bounded set C).
4. For any closed set P satisfying the previous property $\Re\subset P$.

Further, we state that $\Theta_\Sigma = \pi_1\Re$, where $\pi_1 : X\times\Sigma \to X$, $\pi_2 : X\times\Sigma\to\Sigma$ are the projection operators, and that it is a uniform global attractor of the family U_σ.

First, prove that $\Re = \bigcup_{B\in\beta(X)} \omega(B\times\Sigma)$. In fact, for any $C\in\beta(X\times\Sigma)$, $\omega(C)\subset\omega(B\times\Sigma)$, where $B\in\beta(X)$ is such that $C\subset B\times\Sigma$. Hence, $\Re\subset\bigcup_{B\in\beta(X)}\omega(B\times\Sigma)$. Conversely, $\forall B\in\beta(X)$, $B\times\Sigma\in\beta(X\times\Sigma)$, and then $\bigcup_{B\in\beta(X)}\omega(B\times\Sigma)\subset\Re$.

Further, we note that as a consequence of standard theorems for semigroups (see [15, 26]), the continuous semigroup $T(h) : \Sigma\to\Sigma$ has the compact global invariant attractor $\omega(\Sigma) = \bigcap_{t\geq 0}\overline{\bigcup_{h\geq t} T(h)\Sigma} = \bigcap_{t\geq 0}\overline{\gamma_t^+(\Sigma)}$, which is the minimal set attracting any bounded set. We shall check that $\omega(B\times\Sigma)\subset\omega_{0,\Sigma}(B)\times\omega(\Sigma)$, and $\pi_1\omega(B\times\Sigma) = \omega_{0,\Sigma}(B)$, $\pi_2\omega(B\times\Sigma) = \omega(\Sigma)$. On the one hand,

$$\begin{aligned}\omega(B\times\Sigma) &= \bigcap_{t\geq 0}\overline{\bigcup_{s\geq t} G(s,B\times\Sigma)} \subset \bigcap_{t\geq 0}\overline{\bigcup_{s\geq t}(U_+(s,0,B),T(s)\Sigma)}\\ &\subset \bigcap_{t\geq 0}\overline{\gamma_{t,\Sigma}^0(B)\times\gamma_t^+(\Sigma)} = \bigcap_{t\geq 0}\left(\overline{\gamma_{t,\Sigma}^0(B)}\times\overline{\gamma_t^+(\Sigma)}\right)\\ &= \omega_{0,\Sigma}(B)\times\omega(\Sigma).\end{aligned} \tag{4.6}$$

It follows that $\pi_1\omega(B\times\Sigma)\subset\omega_{0,\Sigma}(B)$, $\pi_2\omega(B\times\Sigma)\subset\omega(\Sigma)$. On the other hand, since

$$\begin{aligned}&\mathrm{dist}_{X\times\Sigma}\left(\bigcup_{\sigma\in\Sigma}(U_\sigma(t,0,B),T(t)\sigma),\omega(B\times\Sigma)\right)\\ &\quad = \mathrm{dist}_{X\times\Sigma}(G(t,B\times\Sigma),\omega(B\times\Sigma)) \to 0,\ \text{as } t\to\infty,\end{aligned}$$

it is clear that

$$\mathrm{dist}(U_+(t,0,B),\pi_1\omega(B\times\Sigma)) \to 0,\ \text{as } t\to\infty. \tag{4.7}$$

But in view of Proposition 4.1 for any closed set P having property (4.7), $\omega_{0,\Sigma}(B)\subset P$; hence, $\omega_{0,\Sigma}(B)\subset\pi_1\omega(B\times\Sigma)$ (we note that since $\omega(B\times\Sigma)$ is compact, $\pi_1\omega(B\times\Sigma)$ is closed). In a similar way using the minimality of $\omega(\Sigma)$, we prove that $\omega(\Sigma)\subset\pi_2\omega(B\times\Sigma)$.

Therefore,

$$\pi_1\Re = \pi_1\left(\underset{B\in\beta(X)}{\cup}\omega\left(B\times\Sigma\right)\right) = \underset{B\in\beta(X)}{\cup}\pi_1\omega\left(B\times\Sigma\right) = \underset{B\in\beta(X)}{\cup}\omega_{0,\Sigma}\left(B\right),$$

so that $\pi_1\Re = \Theta_\Sigma$. In the same way, $\pi_2\Re = \omega\left(\Sigma\right)$.

We have to check the conditions of Definition 4.4. For arbitrary $B \in \beta\left(X\right)$, $\sigma \in \Sigma$ and $(t,\tau) \in \mathbf{R}_{+d}$, we have

$$U_\sigma\left(t,\tau,B\right) \subset U_{T(\tau)\sigma}\left(t-\tau,0,B\right) \subset \pi_1 G\left(t-\tau,B\times\Sigma\right),$$

so that Θ_Σ is a uniformly attracting set.

On the other hand, since $\Re \subset G\left(t,\Re\right)$, $\forall t \in \mathbf{R}_+$, we have

$$\Theta_\Sigma = \pi_1\Re \subset \pi_1 G\left(t,\Re\right) \subset \pi_1 G\left(t,\Theta_\Sigma\times\Sigma\right) \subset U_+\left(t,0,\Theta_\Sigma\right).$$

Therefore, Θ_Σ is negatively semiinvariant with respect to $U_+\left(t,0,\cdot\right)$.

It remains to prove the minimality property. Let P be a closed uniformly attracting set. $X\times\Sigma$ is a metric space with the metric

$$d\left(\left(x_1,\sigma_1\right),\left(x_2,\sigma_2\right)\right) = \rho\left(x_1,x_2\right) + r\left(\sigma_1,\sigma_2\right),$$

where r is the metric of Σ. Then for any $B \in \beta\left(X\right)$,

$$\begin{aligned}\operatorname{dist}_{X\times\Sigma}\left(G\left(t,B\times\Sigma\right),P\times\Sigma\right) &= \sup_{x\in B,\sigma\in\Sigma}\operatorname{dist}_{X\times\Sigma}\left(\left(U_\sigma\left(t,0,x\right),T\left(t\right)\sigma\right),P\times\Sigma\right)\\ &\le \operatorname{dist}_{X\times\Sigma}\left(U_+\left(t,0,B\right)\times T\left(t\right)\Sigma,P\times\Sigma\right)\\ &= \sup_{(x,\sigma)\in\left(U_+(t,0,B)\times T(t)\Sigma\right)}\ \inf_{\left(y,\delta\right)\in P\times\Sigma}\left(\rho\left(x,y\right)+r\left(\sigma,\delta\right)\right)\\ &= \operatorname{dist}\left(U_+\left(t,0,B\right),P\right)\underset{t\to+\infty}{\to}0.\end{aligned}$$

Then the set $P\times\Sigma$ is attracting for G, and in view of property 4 of the global attractor $\Re$, $\Re \subset P\times\Sigma$. Hence, $\Theta_\Sigma = \pi_1\Re \subset P$.

It is clear that Θ_Σ is σ-compact, because in view of (4.4) and the inclusion $\omega_{0,\Sigma}\left(B_1\right) \subset \omega_{0,\Sigma}\left(B_2\right)$ if $B_1 \subset B_2$ the set Θ_Σ is a countable union of compact sets. Hence, it is Lindelöf.

In the same way as in Theorem 1.2, we can say that accurate to homeomorphisms $\Theta_\Sigma = \cup_{i=1}^{\infty} D_i$, where $D_i = \left\{(x,i) : x \in \omega_{0,\Sigma}\left(B_i\right)\right\}$, $B_i = \left\{x \in X : \|x\| \le i\right\}$ and D_i are topological spaces with the topology τ_i induced by X. We consider the family $\beta_\oplus = \left\{U \subset \Theta_\Sigma \mid U \cap D_i \in \tau_i \text{ for any } i \ge 1\right\}$, which is a subbase of a topology $\tau_\oplus$ in Θ_Σ, which is called the sum topology. In the space $(\Theta_\Sigma,\tau_\oplus)$, the global attractor is locally compact (see the proof of Theorem 1.2).

Since Θ_Σ is a countable union of compact sets, Baire's theorem implies that $\Theta_\Sigma \ne X$.

Finally, we shall prove the second equality in (4.4). It is a consequence of the following lemma:

Lemma 4.5. *For any* $\tau \in \mathbf{R}_+$*,* $B \in \beta(X)$

$$\omega_{\tau,\Sigma}(B) \subset \omega_{0,\Sigma}(B).$$

Proof. Let $\sigma \in \Sigma$ and $(t,\tau) \in \mathbf{R}_{+d}$. Then

$$U_\sigma(t,\tau,B) \subset U_{T(\tau)\sigma}(t-\tau,0,B) \subset U_+(t-\tau,0,B).$$

Hence, $\forall B \in \beta(X), \forall (t,\tau) \in \mathbf{R}_{+d}$

$$U_+(t,\tau,B) \subset U_+(t-\tau,0,B).$$

Consequently,

$$\gamma^0_{s,\Sigma}(B) = \bigcup_{t' \geq s} U_+\left(t',0,B\right) = \bigcup_{t-\tau \geq s} U_+(t-\tau,0,B),$$

$$\gamma^\tau_{s,\Sigma}(B) = \bigcup_{t \geq s} U_+(t,\tau,B) \subset \gamma^0_{s-\tau,\Sigma}(B).$$

Therefore,

$$\omega_{\tau,\Sigma}(B) = \bigcap_{s \geq \tau} \overline{\gamma^\tau_{s,\Sigma}(B)} \subset \bigcap_{s-\tau \geq 0} \overline{\gamma^0_{s-\tau,\Sigma}(B)} = \omega_{0,\Sigma}(B).$$

□

Definition 4.5. The family of MSP $\{U_\sigma\}$ is called pointwise dissipative if there exists $B_0 \in \beta(X)$ such that $\forall x \in X$

$$\text{dist}\left(U_+(t,0,x), B_0\right) \to 0 \text{ as } t \to +\infty.$$

Definition 4.6. Let X, Y be metric spaces. The multivalued map $F : X \to 2^Y$ is said to be w-upper semicontinuous (w-u.s.c.) at x_0 if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$F(x) \subset O_\varepsilon(F(x_0)), \forall x \in O_\delta(x_0).$$

The map F is w-u.s.c. if it is w-u.s.c. at any $x \in D(F) = \{y \in X : F(x) \neq \emptyset\}$.

If we replace the ε-neighborhood O_ε by an arbitrary one O, then F is called upper semicontinuous.

Remark 4.3. Any upper semicontinuous map is w-upper semicontinuous, the converse being valid if F has compact values [1, p.67].

Lemma 4.6. *A multivalued w-u.s.c. map F with closed values and closed domain $D(F)$ has closed graph.*

Proof. Let $(x_n, y_n) \in Graph(F)$ and $x_n \to x \in D(F)$, $y_n \to y \in Y$. For any ε-neighborhood $O_\varepsilon(F(x))$, there exists n_0 such that $y_n \in O_\varepsilon(F(x))$, $\forall n \geq n_0$. Since the set $F(x)$ is closed, this implies that $y \in F(x)$. □

Theorem 4.2. *Let the conditions of Proposition 4.1 be satisfied, $L1 - L3$ hold, U_σ be pointwise dissipative, and let the map $(x, \sigma) \mapsto U_\sigma(t, 0, x)$ have closed values and be w-u.s.c. for any $t \in \mathbf{R}_+$. Then the family of MSP U_σ has the global compact uniform attractor Θ_Σ.*

Proof. It follows from the definition of the map G that for any fixed $t \in \mathbf{R}_+$, $G(t, \cdot)$ has closed values and is w-upper semicontinuous. Since the semiflow G is uniformly asymptotically upper semicompact in view of Lemma 4.3, Theorem 1.1 implies that the set $\omega(B \times \Sigma)$ is nonempty, compact, negatively semiinvariant and the minimal closed set attracting $B \times \Sigma$. We note that in this theorem, the map $G(t, \cdot)$ is supposed to be upper semicontinuous instead of w-upper semicontinuous, but this property is only used to prove that $G(t, \cdot)$ has closed graph, which is also a consequence of the w-upper semicontinuity (see Lemma 4.6).

In view of Proposition 4.1 for any $B \in \beta(X)$ and $\tau \in \mathbf{R}_+$, the set $\omega_{\tau,\Sigma}(B)$ is nonempty, compact, and the minimal closed set uniformly attracting B. Moreover, it is shown in the proof of Theorem 4.1 (see (4.6)) that $\omega(B \times \Sigma) \subset \omega_{0,\Sigma}(B) \times \omega(\Sigma)$ and $\pi_1 \omega(B \times \Sigma) = \omega_{0,\Sigma}(B)$. Further, Lemma 4.5 implies that $\omega_{0,\Sigma}(B) = \bigcup_{\tau \in \mathbf{R}_+} \omega_{\tau,\Sigma}(B)$.

Let $B_0 \in \beta(X)$ attracts any point $x \in X$. Then, the set $B_0 \times \Sigma$ attracts any $\xi \in X \times \Sigma$, that is

$$\mathrm{dist}_{X \times \Sigma}(G(t, \xi), B_0 \times \Sigma) \to 0, \text{ as } t \to +\infty,$$

and G is pointwise dissipative (see Definition 1.7).

Set $B_1 = O_{\varepsilon_1}(B_0)$ for some fixed $\varepsilon_1 > 0$ and $\Re = \omega(B_1 \times \Sigma)$. We claim that this set is a global compact attractor for G. Since it is compact and negatively semi-invariant, it is only necessary to show that it attracts any bounded set $C \in \beta(X \times \Sigma)$. In fact, it is sufficient to take sets of the type $B \times \Sigma$, where $B \in \beta(X)$. Since the family U_σ is pointwise dissipative for any $\xi \in X \times \Sigma$, there exists $T(\xi)$ such that $G(T, \xi) \subset B_1 \times \Sigma$. Now, by the w-upper semicontinuity of the map U, we can find a neighborhood $O_{\delta(\xi)}(\xi)$ for which $G\left(T, O_{\delta(\xi)}(\xi)\right) \subset B_1 \times \Sigma$. The set $\left\{O_{\delta(\xi)}(\xi) : \xi \in \omega(B \times \Sigma)\right\}$ is an open cover of the compact set $\omega(B \times \Sigma)$. Let $\left\{O_{\delta(\xi_i)}(\xi_i)\right\}_{i=1}^n$ be a finite subcover. Then $O(\omega(B \times \Sigma)) = \cup_{i=1}^n O_{\delta(\xi_i)}(\xi_i)$ is a neighborhood of $\omega(B \times \Sigma)$. If $\varepsilon_2 > 0$, then for any ξ_i, we have

$$\begin{aligned} G\left(t + T(\xi_i), O_{\delta(\xi_i)}(\xi_i)\right) &\subset G\left(t, G\left(T(\xi_i), O_{\delta(\xi_i)}(\xi_i)\right)\right) \\ &\subset G(t, B_1 \times \Sigma) \subset O_{\varepsilon_2}(\omega(B_1 \times \Sigma)), \end{aligned}$$

for any $t \geq T(\varepsilon_2, B_1)$. Hence,

$$G\left(t, O\left(\omega\left(B \times \Sigma\right)\right)\right) \subset O_{\varepsilon_2}\left(\omega\left(B_1 \times \Sigma\right)\right),$$

for any $t \geq \max_i \{T(\xi_i)\} + T(\varepsilon_2, B_1)$. For any $\varepsilon > 0$, there exists $T(\varepsilon, B)$ such that $G(t, B \times \Sigma) \subset O_\varepsilon(\omega(B \times \Sigma))$, $\forall t \geq T(\varepsilon, B)$. The compacity of $\omega(B \times \Sigma)$ implies that $O(\omega(B \times \Sigma))$ contains some neighborhood of the type $O_\varepsilon(\omega(B \times \Sigma))$. Therefore,

$$G\left(t, B \times \Sigma\right) \subset O_{\varepsilon(B)}\left(\omega\left(B \times \Sigma\right)\right) \subset O\left(\omega\left(B \times \Sigma\right)\right), \ \forall t \geq T\left(\varepsilon, B\right),$$

so that

$$G\left(t, B \times \Sigma\right) \subset G\left(t - T\left(\varepsilon, B\right), G\left(T\left(\varepsilon, B\right), B \times \Sigma\right)\right) \subset O_{\varepsilon_2}\left(\omega\left(B_1 \times \Sigma\right)\right),$$

for any $t \geq T(\varepsilon, B) + T(\varepsilon_2, B_1) + \max_i \{T(\xi_i)\}$. This means that the set $\omega(B_1 \times \Sigma)$ attracts any bounded set $C \in X \times \Sigma$ and then it is a global compact attractor.

On the other hand, it is clear from $\pi_1 \omega(B_1 \times \Sigma) = \omega_{0,\Sigma}(B_1)$ that for any $B \in \beta(X)$

$$\operatorname{dist}\left(U_+\left(t, 0, B\right), \omega_{0,\Sigma}\left(B_1\right)\right) \to 0, \text{ as } t \to +\infty. \tag{4.8}$$

The uniformly attracting property follows from the following lemma:

Lemma 4.7. *Property (4.8) is equivalent to*

$$\operatorname{dist}\left(U_+\left(t, \tau, B\right), \omega_{0,\Sigma}\left(B_1\right)\right) \to 0, \text{ as } t \to +\infty,$$

for any $(t, \tau) \in \mathbf{R}_{+d}$.

Proof. Since $U_+(t + \tau, \tau, B) \subset U_+(t, 0, B)$, $\forall (t, \tau) \in \mathbf{R}_{+d}$, $\forall B \in \beta(X)$, we have

$$\operatorname{dist}\left(U_+\left(t + \tau, \tau, B\right), \omega_{0,\Sigma}\left(B_1\right)\right) \leq \operatorname{dist}\left(U_+\left(t, 0, B\right), \omega_{0,\Sigma}\left(B_1\right)\right).$$

□

We note that the minimality property of $\omega_{0,\Sigma}(B)$ implies that $\omega_{0,\Sigma}(B) \subset \omega_{0,\Sigma}(B_1)$, $\forall B \in \beta(X)$. Hence, by (4.4), we have $\omega_{0,\Sigma}(B_1) = \Theta_\Sigma$. From Lemma 4.6 and Theorem 4.1, we obtain that it is a uniform global compact attractor. □

We shall check now that under additional assumptions the global attractor is connected.

Theorem 4.3. *Let the conditions of Theorems 4.1 or 4.2 hold. Also, assume that* Σ *is a connected topological space and that for any* $t \geq 0$, *the map* $\Sigma \times X \ni (\sigma, x) \mapsto U_\sigma(t, 0, x)$ *is upper semicontinuous and has connected values, and* $\Theta_\Sigma \subset B_1$, $B_1 \in \beta(X)$, *where the set* B_1 *is connected. Then the uniform global attractor* Θ_Σ *is connected.*

Proof. Suppose that Θ_Σ is not connected. Then there exist two open sets A_1, A_2 such that $\Theta_\Sigma \cap A_1 \neq \emptyset$, $\Theta_\Sigma \cap A_2 \neq \emptyset$, $\Theta_\Sigma \subset A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$.

Since the map $\Sigma \times X \ni (\sigma, x) \mapsto U_\sigma(t, 0, x)$ is upper semicontinuous and has connected values, $U_\Sigma(t, 0, B_1)$ is a connected set (see Proposition 1.3).

From $\Theta_\Sigma \subset U_+(t, 0, \Theta_\Sigma) \subset U_+(t, 0, B_1)$, we have

$$\{U_+(t, 0, B_1)\} \cap A_1 \neq \emptyset, \quad \{U_+(t, 0, B_1)\} \cap A_2 \neq \emptyset.$$

But $A_1 \cup A_2$ does not cover $U_+(t, 0, B_1)$ for any $t \geq 0$. Thus, there exist $\xi_n \in U_+(t_n, 0, B_1)$, where $t_n \to +\infty$, such that $\xi_\alpha \notin A_1 \cup A_2$. The sequence $\{\xi_n\}$ has a converging subnet and its limit ξ belongs to $\omega_{0,\Sigma}(B_1)$ but does not belong to $A_1 \cup A_2$, which is a contradiction. □

We can extend easily all the results of this section to the case where $\tau \in \mathbf{R}$. We shall state the main theorems for this situation. Let $\mathbf{R}_d = \{(t, s) \in \mathbf{R}^2 : t \geq s\}$.

Definition 4.7. The map $U : \mathbf{R}_d \times X \to P(X)$ is called a multivalued process (MP) if:

1. $U(t, t, \cdot) = Id$ is the identity map.
2. $U(t, \tau, x) \subset U(t, s, U(s, \tau, x))$, $\forall t \geq s \geq \tau, \forall x \in X$, where $U(t, s, U(s, \tau, x)) = \bigcup_{y \in U(s,\tau,x)} U(t, s, y)$.

It is called strict if, moreover, $U(t, \tau, x) = U(t, s, U(s, \tau, x))$.

Consider the family of MP $\{U_\sigma : \sigma \in \Sigma\}$ and define the map $U_+ : \mathbf{R}_d \times X \to P(X)$ by

$$U_+(t, \tau, x) = \bigcup_{\sigma \in \Sigma} U_\sigma(t, \tau, x),$$

which is a multivalued process, as well.

As in the previous case, we say that the family of MP $\{U_\sigma : \sigma \in \Sigma\}$ is uniformly asymptotically upper semicompact if for any $B \in \beta(X)$ and $\tau \in \mathbf{R}$ such that for some $T = T(B, \tau)$, $\gamma^\tau_{T,\Sigma}(B) = \bigcup_{\sigma \in \Sigma} \gamma^\tau_{T,\sigma}(B) \in \beta(X)$, any sequence $\{\xi_n\}$, $\xi_n \in U_{\sigma_n}(t_n, \tau, B)$, $\sigma_n \in \Sigma$, $t_n \to +\infty$, is precompact in X.

Also, the set A is a uniformly attracting set if for any $B \in \beta(X)$ and $\tau \in \mathbf{R}$,

$$\lim_{t \to +\infty} dist(U_+(t, \tau, B), A) = 0.$$

The following results are proved exactly as in the case where $\tau \in \mathbf{R}_+$.

Proposition 4.4. *Let the family of MP $\{U_\sigma : \sigma \in \Sigma\}$ be uniformly asymptotically upper semicompact, and for any $B \in \beta(X)$ and $\tau \in \mathbf{R}$, there exist $T = T(B, \tau)$ such that $\gamma^\tau_{T,\Sigma}(B) = \bigcup_{\sigma \in \Sigma} \gamma^\tau_{T,\sigma}(B) \in \beta(X)$.*

Then $\omega_{\tau,\Sigma}(B) \neq \emptyset$, $\forall B \in \beta(X)$, $\tau \in \mathbf{R}$. Moreover, it is compact in X and

$$dist(U_+(t, \tau, B), \omega_{\tau,\Sigma}(B)) \to 0, \text{ as } t \to +\infty.$$

If P is another closed set such that dist $(U_+(t,\tau,B), P) \underset{t\to+\infty}{\to} 0$, *then* $\omega_{\tau,\Sigma}(B) \subset P$ *(minimality property).*

Proposition 4.5. *Let X be an infinite-dimensional Banach space and the conditions of Proposition 4.4 hold. Then for any $\tau \in \mathbf{R}$, the set* $\Theta_\tau = \underset{B\in\beta(X)}{\cup} \omega_{\tau,\Sigma}(B) \neq X$ *is such that for any* $B \in \beta(X)$,

$$dist\,(U_+(t,\tau,B), \Theta_\tau) \to 0,\ as\ t \to +\infty. \tag{4.9}$$

Moreover, for any closed set Y_τ satisfying (4.9), $\Theta_\tau \subset Y_\tau$.

Let Z be a topological space and $F(\mathbf{R}, Z)$ be some space of functions with values in Z. As in the previous case, we shall further assume the following:

(L1B) $\Sigma \subset F(\mathbf{R}, Z)$ is a compact metric space.
(L2B) On Σ is defined the continuous shift operator $T(h)\sigma(t) = \sigma(t+h)$, $h \in \mathbf{R}$, and $T(h)\Sigma \subseteq \Sigma$.
(L3B) For any $(t,\tau) \in \mathbf{R}_d$, $\sigma \in \Sigma$, $h \in \mathbf{R}$, $x \in X$, we have

$$U_\sigma(t+h, \tau+h, x) \subset U_{T(h)\sigma}(t,\tau,x).$$

Remark 4.4. It follows easily from $L2B - L3B$ that, in fact, $T(h)\Sigma = \Sigma$ and

$$U_\sigma(t+h, \tau+h, x) = U_{T(h)\sigma}(t,\tau,x).$$

Definition 4.8. The set Θ_Σ is a uniform global attractor for the family of MP $\{U_\sigma : \sigma \in \Sigma\}$ if:

1. Θ_Σ is a uniformly attracting set.
2. $\Theta_\Sigma \subset U_+(t,0,\Theta_\Sigma)$, $\forall t \in \mathbf{R}_+$.
3. For any closed uniformly attracting set Y, $\Theta_\Sigma \subset Y$.

Theorem 4.4. *Let X be an infinite-dimensional Banach space, the conditions of Proposition 4.4 be satisfied, and $L1B - L3B$ hold. Suppose that for $t \geq 0$, the map $\Sigma \times X \ni (\sigma, x) \mapsto U_\sigma(t,0,x)$ has closed graph. Then the set*

$$\Theta_\Sigma \overset{def}{=} \underset{B\in\beta(X)}{\cup} \omega_{0,\Sigma}(B) = \underset{\tau\in\mathbf{R}}{\cup}\ \underset{B\in\beta(X)}{\cup} \omega_{\tau,\Sigma}(B) \tag{4.10}$$

is a uniform global attractor and $\Theta_\Sigma \neq X$. It is σ-compact and Lindelöf in X. It is locally compact in the sum topology $\tau_\oplus$.

Remark 4.5. If X is not infinite-dimensional or is a metric space, all the statements remain true except the fact that $\Theta_\Sigma \neq X$.

Remark 4.6. As in view of Remark 4.4 we have $U_\sigma(t,\tau,x) = U_{T(\tau)\sigma}(t-\tau, 0, x)$, we obtain in fact that the map $\Sigma \times X \ni (\sigma, x) \mapsto U_\sigma(t,\tau,x)$ has closed graph for any $t \geq \tau$.

As before, we say that the family of MP $\{U_\sigma\}$ is pointwise dissipative if there exists $B_0 \in \beta(X)$ such that

$$dist\,(U_+(t,0,B), B_0) \to 0 \text{ as } t \to +\infty.$$

Theorem 4.5. *Let the conditions of Proposition 4.4 be satisfied, $L1B - L3B$ hold, U_σ be pointwise dissipative, and let the map $(x,\sigma) \mapsto U_\sigma(t,0,x)$ have closed values and be w-u.s.c. for any $t \in \mathbf{R}_+$. Then the family of MP U_σ has the global compact uniform attractor Θ_Σ.*

Theorem 4.6. *Let the conditions of Theorems 4.4 or 4.5 hold. Also, assume that Σ is a connected topological space and that for any $t \geq 0$, the map $\Sigma \times X \ni (\sigma, x) \mapsto U_\sigma(t,0,x)$ is upper semicontinuous and has connected values, and $\Theta_\Sigma \subset B_1$, $B_1 \in \beta(X)$, where the set B_1 is connected. Then the global attractor Θ_Σ is connected.*

We will prove also that the attractor is invariant if the processes U_σ are strict.

Lemma 4.8. *Let the conditions of Theorems 4.4 or 4.5 hold. Also, assume that the family of MP U_σ is strict. Then*

$$\Theta_\Sigma = U_+(t,0,\Theta_\Sigma) \textit{ for all } t \geq 0.$$

Proof. Let us prove first that $\omega_{\tau,\Sigma}(B) \subset U_+(t,\tau,\omega_{\tau,\Sigma}(B))$ for all $t \geq \tau$ and $B \in \beta(X)$. Let $\xi \in \omega_{\tau,\Sigma}(B)$. Then there exists a sequence $\xi_n \in U_+(t_n,\tau,B)$ such that $\xi_n \to \xi$ as $t_n \to +\infty$. Using $L3R$ for $t \geq \tau$, we have

$$\begin{aligned} U_{\sigma_n}(t_n,\tau,B) &\subset U_{\sigma_n}(t_n, t_n - t + \tau, U_{\sigma_n}(t_n - t + \tau, \tau, B)) \\ &\subset U_{T(t_n-t)\sigma_n}(t,\tau,U_{\sigma_n}(t_n - t + \tau,\tau,B)), \end{aligned}$$

and, therefore, $\xi_n \in U_{T(t_n-t)\sigma_n}(t,\tau,\zeta_n)$, where $\zeta_n \in U_{\sigma_n}(t_n - t + \tau, \tau, B)$. As Σ is compact, we can assume that $T(t_n - t)\sigma_n \to \sigma$. Then we can consider that $\xi_n \to \xi$, $\zeta_n \to \zeta \in \omega_{\tau,\Sigma}(\tau)$. Since in view of Remark 4.6 $\Sigma \times X \ni (\sigma, x) \mapsto U_\sigma(t,\tau,x)$ has closed graph, $\xi \in U_\sigma(t,\tau,\omega_{\tau,\Sigma}(B))$. Hence, $\omega_{\tau,\Sigma}(B) \subset U_+(t,\tau,\omega_{\tau,\Sigma}(B))$.

Then, using Remark 4.4, we have

$$\omega_{0,\Sigma}(B) \subset U_+(t,0,\omega_{0,\Sigma}(B)) = U_+(t+\tau,\tau,\omega_{0,\Sigma}(B)),$$

for all $\tau \in \mathbf{R}$, $t \in \mathbf{R}_+$. Hence, for arbitrary $\xi \in \omega_{0,\Sigma}(B)$ and $\tau \geq p$, it follows that $\xi \in U_\sigma(\tau, p, \omega_{0,\Sigma}(B))$ for some $\sigma \in \Sigma$. Then for any $t \geq \tau \geq p$,

$$U_\sigma(t,\tau,\xi) \subset U_\sigma(t,\tau,U_\sigma(\tau,p,\omega_{0,\Sigma}(B))) = U_\sigma(t,p,\omega_{0,\Sigma}(B)).$$

Therefore, by $L2R - L3R$, we have

$$U_+(t,\tau,\omega_{0,\Sigma}(B)) \subset U_+(t,p,\omega_{0,\Sigma}(B)) \subset U_+(t+\tau-p,\tau,\omega_{0,\Sigma}(B)).$$

Then for all $t \geq s \geq \tau$,

$$U_+(t,\tau,\omega_{0,\Sigma}(B)) \subset cl_X(\bigcup_{k\geq s} U_+(k,\tau,\omega_{0,\Sigma}(B)))$$

$$\subset \bigcap_{s\geq\tau} cl_X(\bigcup_{k\geq s} U_+(k,\tau,\omega_{0,\Sigma}(B))) = \omega_{\tau,\Sigma}(\omega_{0,\Sigma}(B)) \subset \Theta_\Sigma.$$

Thus, $U_+(t,0,\Theta_\Sigma) \subset \Theta_\Sigma$ and the theorem are proved. □

4.2 Global Attractors for Nonautonomous Differential Inclusions

In this section, we shall consider multivalued semiprocesses generated by nonautonomous evolution inclusions.

4.2.1 Abstract Setting: Construction of the Multivalued Semiprocess

Let X, X^* be a real separable Banach space and its dual, with norms $\|\cdot\|_X, \|\cdot\|_{X^*}$, respectively, and pairing denoted by $\langle\cdot,\cdot\rangle$, Z be a complete metric space and $\mathscr{F}(\mathbf{R}_+, Z)$ some functional space. Consider the evolution inclusion

$$\begin{cases} \frac{du(t)}{dt} \in A\left(u\left(t\right)\right) + F_{\sigma(t)}\left(u\left(t\right)\right), \ t \in [\tau, T], \\ u\left(\tau\right) = u_\tau, \end{cases} \tag{4.11}$$

where $T > \tau \geq 0$, $\sigma(\cdot) \in \Sigma \subset \mathscr{F}(\mathbf{R}_+, Z)$ and $A : D(A) \subset X \to 2^X$, $F_{\sigma(\cdot)}(\cdot) : \mathbf{R}_+ \times X \to 2^X$, are multivalued maps satisfying:

(A) The operator A is m-dissipative, that is, $\forall y_1, y_2 \in D(A)$, $\forall \xi_i \in A(y_i)$, $i = 1, 2$, $\exists j(y_i, \xi_i) \in J(y_1 - y_2)$ such that

$$< \xi_1 - \xi_2, j > \leq 0,$$

and $Im(A - \lambda I) = X$, $\forall \lambda > 0$, where $J : X \to 2^{X^*}$ is the duality map defined by

$$J(y) = \{\xi \in X^* \mid < y, \xi >= \|y\|_X^2 = \|\xi\|_{X^*}^2\}, \ \forall y \in X.$$

(G1) $\forall \sigma(\cdot) \in \Sigma$, $F_{\sigma(\cdot)}(\cdot) : \mathbf{R}_+ \times X \to Cb(X)$, where $Cb(X)$ is the set of all nonempty, closed, bounded subsets of X.

(G2) $\forall x \in \overline{D(A)}$, $\sigma \in \Sigma$, the map $t \mapsto F_{\sigma(t)}(x)$ is measurable, and for any $\sigma \in \Sigma$, $T > \tau \geq 0$, there exists $k(\cdot) \in L_1([\tau, T])$ such that $\forall x_1, x_2 \in \overline{D(A)}$

$$\text{dist}_H\left(F_{\sigma(t)}(x_1), F_{\sigma(t)}(x_2)\right) \leq k(t)\|x_1 - x_2\|_X, \text{ a.e. } t \in (\tau, T).$$

(G3) For any $T > \tau \geq 0$, $\sigma \in \Sigma$, there exists $x \in \overline{D(A)}$ and $n : [\tau, T] \to \mathbf{R}_+$, $n(\cdot) \in L_1([\tau, T])$, such that

$$\left\|F_{\sigma(t)}(x)\right\|_+ \leq n(t), \text{ a.e. } t \in (\tau, T),$$

where $\|K\|_+ = \sup_{y \in K}\|y\|_X$.

We assume also that for any $h \in \mathbf{R}_+$, the shift operator $T(h) : \Sigma \to \Sigma$, $T(h)\sigma(s) = \sigma(s+h)$, is defined.

Definition 4.9. The continuous function $u_\sigma(\cdot) \in C([\tau, T], X)$ is called an integral solution of (4.11) if $u_\sigma(\tau) = u_\tau$ and there exists $l(\cdot) \in L_1([\tau, T], X)$ such that $l(t) \in F_{\sigma(t)}(u_\sigma(t))$, a.e. on (τ, T), and $\forall \xi \in D(A)$, $\forall v \in A(\xi)$,

$$\|u_\sigma(t) - \xi\|_X^2 \leq \|u_\sigma(s) - \xi\|_X^2 + 2\int_s^t < l(r) + v, u_\sigma(r) - \xi >_+ dr, \; t \geq s, \quad (4.12)$$

where $< \eta, y >_+ = \sup_{j \in J(y)} < \eta, j >$.

Supposing that conditions $A, G1 - G3$ hold for any $u_\tau \in \overline{D(A)}$, there exists at least one integral solution of (4.11) for any $T > \tau \geq 0$ [40, Theorem 3.1]. We shall denote this solution by $u_\sigma(\cdot) = I(u_\tau)\, l(\cdot)$. We note that $u_\sigma(t) \in \overline{D(A)}$ for all $t \geq 0$. For a fixed $\sigma \in \Sigma$, let $\mathscr{D}_{\sigma,\tau}(x, T)$ be the set of all integral solutions defined on $[0, T]$ corresponding to the initial condition $u(\tau) = x$. Denote

$$\mathscr{D}_{\sigma,\tau}(x) = \cup_{T>0}\mathscr{D}_{\sigma,\tau}(x, T).$$

For any integral solutions $u_\sigma(\cdot) = I(u_\tau)\, l_1(\cdot)$, $v_\sigma(\cdot) = I(v_\tau)\, l_2(\cdot)$, the following inequality holds

$$\|u_\sigma(t) - v_\sigma(t)\|_X \leq \|u_\sigma(s) - v_\sigma(s)\|_X + \int_s^t \|l_1(r) - l_2(r)\|_X\, dr, \; t \geq s. \quad (4.13)$$

In the sequel, we shall write u instead of u_σ for simplicity of notation. We shall define the map $U_\sigma : \mathbf{R}_{+d} \times \overline{D(A)} \to P\left(\overline{D(A)}\right)$ by

$$U_\sigma(t, \tau, x) = \{z : \exists u(\cdot) \in \mathscr{D}_{\sigma,\tau}(x), u(t) = z\}.$$

Proposition 4.6. *For each $\sigma \in \Sigma$, $(t,\tau) \in \mathbf{R}_{+d}$, $h \in \mathbf{R}_+$, $\tau \le s \le t$, $x \in \overline{D(A)}$*

$$U_\sigma(t,s,U_\sigma(s,\tau,x)) = U_\sigma(t,\tau,x),$$
$$U_{T(h)\sigma}(t,\tau,x) = U_\sigma(t+h,\tau+h,x).$$

Hence, U_σ is a multivalued semiprocess for each $\sigma \in \Sigma$ and condition L3 holds.

Proof. Given $z \in U_\sigma(t_1+t_2,\tau,x)$, we have to prove that $z \in U_\sigma(t_1+t_2,t_2,U_\sigma(t_2,\tau,x))$. Take $y(\cdot) \in \mathscr{D}_{\sigma,\tau}(x)$ such that $y(\tau) = x$ and $y(t_1+t_2) = z$. Clearly, $y(t_2) \in U_\sigma(t_2,\tau,x)$. Then if we define $z(t) = y(t)$ for $t \ge t_2$, we have that $z(t_2) = y(t_2)$ and obviously $z(\cdot) \in \mathscr{D}_{\sigma,t_2}(y(t_2))$. Consequently, $z = z(t_1+t_2) \in U_\sigma(t_1+t_2,t_2,U_\sigma(t_2,\tau,x))$.

Conversely, given $z \in U_\sigma(t_1+t_2,t_2,U_\sigma(t_2,\tau,x))$, we have to prove that $z \in U_\sigma(t_1+t_2,\tau,x)$. There exist $y_1(\cdot) \in \mathscr{D}_{\sigma,\tau}(x)$, $y_2(\cdot) \in \mathscr{D}_{\sigma,t_2}(y_1(t_2))$, such that $z = y_2(t_1+t_2)$. Define

$$y(t) = \begin{cases} y_1(t), \text{ if } \tau \le t \le t_2, \\ y_2(t), \text{ if } t_2 \le t \le t_1+t_2, \end{cases}$$

$$l(t) = \begin{cases} l_1(t), \text{ if } \tau \le t \le t_2, \\ l_2(t), \text{ if } t_2 < t \le t_1+t_2, \end{cases}$$

where $y_1(\cdot) = I(x)\,l_1(\cdot)$, $y_2(\cdot) = I(y_1(t_2))\,l_2(\cdot)$. Clearly, $l(t) \in F_{\sigma(t)}(y(t))$, a.e. on (τ,t_1+t_2). We claim that $y(\cdot) \in \mathscr{D}_{\sigma,\tau}(x)$. Indeed, firstly $y(\tau) = x$. Secondly, we have to check that (4.12) holds. The cases $s \le t \le t_2$, $t_2 \le s \le t$ are straightforward. If $s \le t_2 \le t$, then

$$\begin{aligned}\|y(t)-\xi\|_X^2 = \|y_2(t)-\xi\|_X^2 &\le \|y_2(t_2)-\xi\|_X^2 \\ &+ 2\int_{t_2}^{t} < l_2(r)+v, y_2(r)-\xi >_+ dr \\ &\le \|y_1(s)-\xi\|_X^2 + 2\int_s^{t_2} < l_1(r)+v, y_1(r)-\xi >_+ dr \\ &+ 2\int_{t_2}^{t} < l_2(r)+v, y_2(r)-\xi >_+ dr \\ &= \|y(s)-\xi\|_X^2 + 2\int_s^t < l(r)+v, y(r)-\xi >_+ dr.\end{aligned}$$

Hence, $z = y_2(t_1+t_2) = y(t_1+t_2) \in U_\sigma(t_1+t_2,\tau,x)$.

Consider now the second equality. Given $z \in U_\sigma(t_1+h,\tau+h,x)$, where $h \in \mathbf{R}_+$, there exists $y(\cdot) \in \mathscr{D}_{\sigma,\tau+h}(x)$ such that $z = y(t_1+h)$. Let $w(t) = y(t+h)$, $l_w(t) = l(t+h)$, where $l(t) \in F_{\sigma(t)}(y(t))$, a.e. on $(\tau+h,t_1+h)$, $y(\cdot) = I(x)\,l(\cdot)$, so that $l_w(t) \in F_{\sigma(t+h)}(y(t+h)) = F_{T(h)\sigma(t)}(w(t))$, a.e. on

(τ, t_1), and $w(\tau) = x$. We can show that $w(\cdot) \in \mathscr{D}_{T(h)\sigma,\tau}(x)$ as follows

$$\|w(t) - \xi\|_X^2 = \|y(t+h) - \xi\|_X^2 \le \|y(s+h) - \xi\|_X^2$$
$$+ 2\int_{s+h}^{t+h} < l(r) + v, y(r) - \xi >_+ dr$$
$$= \|w(s) - \xi\|_X^2 + 2\int_s^t < l_w(r) + v, w(r) - \xi >_+ dr.$$

Therefore, $z = w(t_1) \in U_{T(h)\sigma}(t_1, \tau, x)$.

Finally, let $z \in U_{T(h)\sigma}(t_1, \tau, x)$. Then $z = y(t_1)$, where $y(\cdot) \in \mathscr{D}_{T(h)\sigma,\tau}(x)$. Define $w(t) = y(t-h)$, $l_w(t) = l(t-h)$, where $l(t) \in F_{\sigma(t+h)}(y(t))$, a.e. on (τ, t_1), $y(\cdot) = I(x)l(\cdot)$. Hence, $l_w(t) \in F_{\sigma(t)}(y(t-h)) = F_{\sigma(t)}(w(t))$, *a.e.* on $(\tau + h, t_1 + h)$. As before, we can prove that $w(\cdot) \in \mathscr{D}_{\sigma,\tau+h}(x)$; hence, $z = w(t_1 + h) \in U_\sigma(t_1 + h, \tau + h, x)$. □

Remark 4.7. It follows from the proof of the preceding lemma that any integral solution can be extended to the whole-time interval $[\tau, +\infty)$. Then if we define $\mathscr{D}_{\sigma,\tau}(x)$ as the set of functions $u : [\tau, +\infty) \to \overline{D(A)}$, which are integral solutions on every interval $[\tau, T]$ with $u(\tau) = x$, then the operator U_σ does not change. In the sequel, we shall use this definition of $\mathscr{D}_{\sigma,\tau}(y_0)$.

Consider the particular case where $X = \varphi(\Omega, \Theta)$ is a real separable Banach space of functions defined on the set Ω and taking values in the normed space Θ. Let us define the map $F_{\sigma_0} : \mathbf{R}_+ \times X \to 2^X$ by

$$F_{\sigma_0(t)}(u) = F^{\sigma_0}(t, u) + g_0(t),$$

where $g_0(\cdot) \in L_2^{loc}(\mathbf{R}_+, X)$ and $F^{\sigma_0} : \mathbf{R}_+ \times X \to 2^X$ is generated by the multivalued map $f_0 : \mathbf{R}_+ \times \Theta \to K(\Theta)$ (i.e., it has compact values) as follows

$$F^{\sigma_0}(t, u) = \{y \in X : y(x) \in f_0(t, u(x)) \text{ for } x \in \Omega\}.$$

Moreover, $K(\Theta)$ is endowed with the Hausdorff metric and $f_0 \in C(\mathbf{R}_+, C(\Theta, K(\Theta)))$.

Lemma 4.9. *Let Θ be a locally compact, Lindelöf, Banach space. Then $C(\Theta, K(\Theta))$ is a complete metrizable space.*

Proof. Under the above conditions, there exists a countable cover $\{V_i : i = 1, 2, \ldots\}$ of the space Θ by open sets such that their closure is compact. For each $i = 1, 2 \ldots$ set $K_i = \overline{\cup_{s=1}^i V_s}$. The sequence of compact sets K_i has the following properties:

1. $K_1 \subset K_2 \subset \ldots \subset K_n \subset \ldots$.
2. For any compact set $K \subset \Theta$, there exists i such that $K \subset K_i$.

Hence, in $C(\Theta, K(\Theta))$ is defined the topology of uniform convergence on $\{K_i\}$, that is, $F_n \to F$ if and only if for any i

$$\rho_i (F_n, F) = \sup_{x \in K_i} \mathrm{dist}_H (F_n (x), F (x)) \to 0, \text{ as } n \to +\infty.$$

This convergence can be realized by the metric

$$\rho (F_1, F_2) = \sum_{i=1}^{\infty} \alpha_i \frac{\rho_i (F_1, F_2)}{1 + \rho_i (F_1, F_2)},$$

where $\alpha_i > 0$ and $\sum_{i=1}^{\infty} \alpha_i < \infty$.

$K (\Theta)$ is a complete metric space. Hence, $C (\Theta, K (\Theta))$ is also complete. □

Let $\mathscr{M} \subset C (\Theta, K (\Theta))$ be a closed subset (and then a complete metrizable space with the metric of $C (\Theta, K (\Theta)))$ and $\sigma_0 (t) = (f_0 (t, \cdot), g_0 (t)) \in Z = \mathscr{M} \times X$. Assume that $\sigma_0 (t + h) = (f_0 (t + h, \cdot), g_0 (t + h)) \in Z$, $\forall h \in \mathbf{R}_+$. Let Σ be the hull $\mathscr{H}_+ (\sigma_0)$ of the map $\sigma_0 (t)$ in the space $C (\mathbf{R}_+, \mathscr{M}) \times L_{2,w}^{loc} (\mathbf{R}_+, X) = \mathscr{F} (\mathbf{R}_+, Z)$, that is,

$$\Sigma = cl_{C(\mathbf{R}_+,\mathscr{M})} \{f_0 (t + h) : h \in \mathbf{R}_+\} \times cl_{L_{2,w}^{loc}(\mathbf{R}_+,X)} \{g (t + h) : h \in \mathbf{R}_+\},$$

where cl_Y denotes the closure in the space Y and $L_{2,w}^{loc} (\mathbf{R}_+, X)$ is the space $L_2^{loc} (\mathbf{R}_+, X)$ with the weak topology.

It is obvious that the shift operator $T (h)$ is continuous in $C (\mathbf{R}_+, \mathscr{M}) \times L_{2,w}^{loc} (\mathbf{R}_+, X)$. Further, we shall give conditions providing Σ to be a compact positively invariant set.

Lemma 4.10. *[11, Proposition 6.1] The hull $\mathscr{H}_+ (f_0) = cl_{C(\mathbf{R}_+,\mathscr{M})}\{f_0(t + h) : h \in \mathbf{R}_+\}$ of f_0 is compact in $C (\mathbf{R}_+, \mathscr{M})$ if and only if:*

1. *The set $\{f_0 (t) : t \in \mathbf{R}_+\}$ is precompact in $\mathscr{M}$.*
2. *$f_0 (t)$ is uniformly continuous in $\mathbf{R}_+$.*

Lemma 4.11. *Let conditions 1–2 of Lemma 4.10 hold and let*

$$\sup_{t \geq 0} \int_t^{t+1} \|g_0 (\tau)\|_X^2 \, d\tau < +\infty. \tag{4.14}$$

Then the hull Σ is compact.

Proof. Condition (4.14) implies that the hull

$$\mathscr{H}_+ (g_0) = cl_{L_{2,w}^{loc}(\mathbf{R}_+,X)} \{g_0 (t + h) : h \in \mathbf{R}_+\}$$

of g_0 is compact in $L_{2,w}^{loc} (\mathbf{R}_+, X)$ (see [11, p.931]). From this and Lemma 4.10, we obtain that Σ is compact. □

Lemma 4.12. *For any $h \in \mathbf{R}_+$, $T (h) \Sigma \subset \Sigma$.*

Proof. Let $y = (y_1, y_2) \in \Sigma$. Then there exists h_n such that $f_0(t+h_n) \to y_1$ in $C(\mathbf{R}_+, \mathscr{M})$, $g_0(t+h_n) \to y_2$ in $L^{loc}_{2,w}(\mathbf{R}_+, X)$, as $n \to \infty$. It is clear that

$$T(h)(f_0(t+h_n), g(t+h_n)) = (f_0(t+h_n+h), g_0(t+h_n+h)) \in \Sigma$$

and in view of the continuity of $T(h)$,

$$(f_0(t+h_n+h), g_0(t+h_n+h)) \to (y_1(t+h), y_2(t+h)),$$

in $C(\mathbf{R}_+, \mathscr{M}) \times L^{loc}_{2,w}(\mathbf{R}_+, X)$, so that $T(h)\, y \in \Sigma$. Hence, $T(h)\,\Sigma \subset \Sigma$. □

Now, the maps $F_\sigma : \mathbf{R}_+ \times X \to 2^X$ are defined by

$$F_{\sigma(t)}(u) = F^\sigma(t,u) + g_\sigma(t),$$

where

$$F^\sigma(t,u) = \{y \in X : y(x) \in f_\sigma(t, u(x)) \text{ for } x \in \Omega\}$$

and $(f_\sigma, g_\sigma) \in \Sigma$.

As a consequence of Theorems 4.1 and 4.2, Proposition 4.6, and Lemmas 4.11 and 4.12, we obtain the following abstract result:

Theorem 4.7. *Let Θ be a locally compact, Lindelöf, Banach space. Suppose that conditions $A, G1-G3$ are satisfied and that for any $t \in \mathbf{R}_+$, the map $(\sigma, x) \mapsto U_\sigma(t,0,x)$ has closed graph. Let the family of MSP $\{U_\sigma : \sigma \in \Sigma\}$ be uniformly asymptotically upper semicompact, and for any $B \in \beta(X)$ and $\tau \in \mathbf{R}_+$, there exist $T = T(B,\tau)$ such that $\gamma^\tau_{T,\Sigma}(B) = \bigcup_{\sigma\in\Sigma} \gamma^\tau_{t,\sigma}(B) \in \beta(X)$. Assume also that the conditions of Lemma 4.11 hold. Then the family U_σ has the uniform global attractor Θ_Σ defined by (4.10).*

Moreover, if the family U_σ is pointwise dissipative and for any $t \in \mathbf{R}_+$, the map $(\sigma, x) \mapsto U_\sigma(t,0,x)$ has closed values and is w-upper semicontinuous, then Θ_Σ is compact.

4.2.2 Global Attractors of Nonautonomous Reaction-Diffusion Inclusions

Let $C_v(\mathbf{R})$ be the set of all nonempty, compact, convex subsets of $\mathbf{R}$ and $\Omega \subset \mathbf{R}^n$ be a bounded open subset with smooth boundary $\partial\Omega$. Consider the parabolic inclusion

$$\begin{cases} \frac{\partial u}{\partial t} - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right) \in f(t,u) + g(t), \text{ on } \Omega \times (\tau, T), \\ u\mid_{\partial\Omega} = 0, \\ u\mid_{t=\tau} = u_\tau, \end{cases} \tag{4.15}$$

where $p \geq 2$, $f : \mathbf{R}_+ \times \mathbf{R} \to C_v(\mathbf{R})$, $g \in L_\infty(\mathbf{R}_+, L_2(\Omega))$ and the following conditions hold:

(F1) There exists $C \geq 0$ such that $\forall t \in \mathbf{R}_+, \forall u, v \in \mathbf{R}$,

$$\operatorname{dist}_H(f(t,u), f(t,v)) \leq C|u-v|.$$

(F2) For any $t, s \in \mathbf{R}_+$ and $u \in \mathbf{R}$,

$$\operatorname{dist}_H(f(t,u), f(s,u)) \leq l(|u|)\,\alpha(|t-s|),$$

where α is a continuous function such that $\alpha(t) \to 0$, as $t \to 0^+$, and l is a continuous nondecreasing function. Moreover, there exist $K_1, K_2 \geq 0$ such that

$$|l(u)| \leq K_1 + K_2|u|, \ \forall u \in \mathbf{R}.$$

(F3) $\exists D \in \mathbf{R}_+, v_0 \in \mathbf{R}$ for which

$$|f(t,v_0)|_+ \leq D, \ \forall t \in \mathbf{R}_+,$$

where $|f(t,v_0)|_+ = \sup_{\zeta \in f(t,v_0)} |\xi|$.

(F4) If $p = 2$, then there exist $\epsilon > 0$ and $M \geq 0$ such that $\forall u \in \mathbf{R}, \forall t \in \mathbf{R}_+, \forall y \in f(t,u)$

$$yu \leq (\lambda_1 - \epsilon)u^2 + M,$$

where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

Lemma 4.13. *There exist* $D_1, D_2 \geq 0$ *such that* $\forall u \in \mathbf{R}, \forall t \in \mathbf{R}_+, \forall y \in f(t,u)$,

$$|y| \leq D_1 + D_2|u|.$$

Proof. Since $f(t,v_0)$ is compact, for any $u \in \mathbf{R}$, $t \in \mathbf{R}_+$, $y \in f(t,u)$, there exists $y_0 \in f(t,v_0)$ such that $\operatorname{dist}(y, f(t,v_0)) = |y - y_0|$. Using $F1$ and $F3$, we obtain

$$|y| \leq D + C|u| + C|v_0|.$$

□

Following the notation of the previous section $\Theta = \mathbf{R}$, $X = \varphi(\Omega,\Theta) = L_2(\Omega)$ and $f_0(t,\cdot) = f(t,\cdot) : \mathbf{R} \to K(\mathbf{R})$, $g_0 = g \in L_\infty(\mathbf{R}_+, L_2(\Omega)) \subset L_2^{loc}(\mathbf{R}_+, L_2(\Omega))$. Let W be the space $C_v(\mathbf{R})$ endowed with the Hausdorff metric $\rho(x,y) = \operatorname{dist}_H(x,y)$. The space $W \subset K(\mathbf{R})$ is complete.

For any $\psi \in W$, let $|\psi|_+ = \max_{y\in\psi} |y|$. Define also the space

$$\mathscr{M} = \left\{\psi \in C(\mathbf{R}, W) : |\psi(v)|_+ \leq D_1 + D_2|v|\right\}.$$

The constants D_1, D_2 are taken from Lemma 4.13. If in Lemma 4.9 take $K_i = [-R_i, R_i]$, where $R_1 < R_2 < \ldots < R_n < ..$, with $R_n \to \infty$, we have that $\psi^m \to \psi$ if and only if

$$\max_{|v|\le R} \mathrm{dist}_H \left(\psi^m (v), \psi (v)\right) \to 0, \text{ as } m \to \infty, \forall R > 0.$$

The space $\mathscr{M} \subset C(\mathbf{R}, K(\mathbf{R}))$ is complete. Indeed, it is sufficient to check that $\mathscr{M}$ is closed. If $\psi^m \to \psi$, then it is clear from Lemma 4.9 that $\psi \in C(\mathbf{R}, K(\mathbf{R}))$. Since W is complete, $\psi(v)$ is convex, and then $\psi \in C(\mathbf{R}, W)$. On the other hand, if we fix $v \in \mathbf{R}$, we get

$$|\psi^m (v)|_+ \le D_1 + D_2 |v|, \forall m.$$

Hence, $\forall \epsilon > 0$, $\forall y \in \psi(v)$, $\exists y^m \in \psi^m(v)$ such that

$$|y - y^m| < \epsilon$$

and then

$$|y| \le D_1 + D_2 |v| + \epsilon.$$

Since y, ε are arbitrary, we have $|\psi(v)|_+ \le D_1 + D_2 |v|$, so that $\psi \in \mathscr{M}$.

Lemma 4.14. *In the space W, each bounded set is precompact.*

Proof. It is clear that a sequence $I_n = [a_n, b_n]$ is bounded in W if and only if there exists $R > 0$ such that

$$|y| \le R, \forall y \in I_n, \forall n.$$

If I_n is bounded, there exists a subsequence I_n' such that $a_n' \to a$, $b_n' \to b$. Therefore, $\alpha a_n' + (1-\alpha) b_n' \to \alpha a + (1-\alpha) b$, $\forall \alpha \in [0, 1]$. Hence, if $I = [a, b]$, then

$$\mathrm{dist}_H \left(I_n', I\right) \to 0, \text{ as } n \to +\infty.$$

□

Let $\Phi \subset \mathscr{M}$ be the set

$$\Phi = \left\{\psi \in \mathscr{M} : \mathrm{dist}_H \left(\psi(u), \psi(v)\right) \le C |u - v|, \forall u, v \in \mathbf{R}\right\},$$

where C is defined in $F1$.

Lemma 4.15. *The set Φ is compact.*

Proof. Let π_R be the projector of $C([-K, K], W)$ onto $C([-R, R], W)$, where $R < K$. Let $\{\psi_n\} \subset \Phi$ be an arbitrary sequence. It is clear that these functions are equicontinuous. On the other hand, in view of the definition of the space $\mathscr{M}$, there exists $D(R)$ such that

$$|\psi_n (v)|_+ \le D(R), \forall v \in [-R, R].$$

By Lemma 4.14 and Ascoli–Arzela theorem, $\{\psi_n\}$ is precompact in $C([-R,R], W)$. Hence, we can take a converging subsequence $\psi_{1n} \to \psi_1$ in $C([-R,R], W)$. Further, we take a subsequence $\{\psi_{2n}\} \subset \{\psi_{1n}\}$ converging to ψ_2 in $C([-2R, 2R], W)$. It is evident that $\pi_R \psi_2 = \psi_1$. In the same way, we can construct the chain of subsequences $\{\psi_{1n}\} \supset \{\psi_{2n}\} \supset \ldots \supset \{\psi_{jn}\} \ldots$ such that $\psi_{jn} \to \psi_j$ in $C([-jR, jR], X)$ and $\pi_{(j-1)R}\psi_j = \psi_{j-1}$, $\forall j \geq 2$. We define a map $\psi \in C(\mathbf{R}, W)$ such that

$$\pi_{jR}\psi = \psi_j, \forall j \geq 1.$$

The function ψ belongs to Φ. Indeed, since $\psi \subset \mathscr{M}$, it is sufficient to show that ψ_j satisfies the Lipschitz property for any $j \geq 1$. Let $\epsilon > 0$ be arbitrary and n be such that $\operatorname{dist}_H \left(\psi_{jn}(v), \psi_j(v)\right) \leq \epsilon$, $\forall v \in [-jR, jR]$. For any $u, v \in [-jR, jR]$, we have

$$\begin{aligned} &\operatorname{dist}_H \left(\psi_j(u), \psi_j(v)\right) \\ &\leq \operatorname{dist}_H \left(\psi_j(u), \psi_{jn}(u)\right) + \operatorname{dist}_H \left(\psi_{jn}(u), \psi_{jn}(v)\right) + \operatorname{dist}_H \left(\psi_j(v), \psi_{jn}(v)\right) \\ &\leq 2\epsilon + C\,|u - v|\,. \end{aligned}$$

Since ε is arbitrary small, ψ_j satisfies the Lipschitz property. Then $\psi \in \Phi$, and finally, we can see that the diagonal subsequence $\{\psi_{jj}\}$ converges to ψ in $\mathscr{M}$. □

Recall that the hull of $f \in C(\mathbf{R}_+, \mathscr{M})$ is defined by

$$\mathscr{H}_+(f) = cl_{C(\mathbf{R}_+,\mathscr{M})}\{f(t+h) : h \geq 0\}.$$

Definition 4.10. The function $f \in C(\mathbf{R}_+, \mathscr{M})$ is said to be translation compact if its hull $\mathscr{H}_+(f)$ is compact in $C(\mathbf{R}_+, \mathscr{M})$.

Lemma 4.16. *The function f is translation compact.*

Proof. In view of $F1$ and Lemma 4.13, $f(s,\cdot) \in \Phi$, $\forall s \geq 0$. Hence, the set

$$\{f(t) : t \in \mathbf{R}_+\}$$

is precompact by Lemma 4.15. On the other hand, $F2$ implies that

$$\rho_{\mathscr{M}}(f(t), f(s)) = \sum_{i=1}^{\infty} \alpha_i \frac{\max_{|v| \leq R_i} \operatorname{dist}_H(f(t,v), f(s,v))}{1 + \max_{|v| \leq R_i} \operatorname{dist}_H(f(t,v), f(s,v))} \leq \epsilon,$$

if $|t - s| \leq \delta(\epsilon)$. Hence, $f(s)$ is uniformly continuous. By Lemma 4.10, f is translation compact. □

Since $g \in L_\infty(\mathbf{R}_+, L_2(\Omega))$, we have that (4.14) holds, so that in view of Lemma 4.11, the symbol $\sigma_0(t) = (f(t,\cdot), g(t))$ is translation compact in the space

$$C\left(\mathbf{R}_+, \mathscr{M}\right) \times L_{2,w}^{loc}\left(\mathbf{R}_+, L_2\left(\Omega\right)\right).$$

The hull of this symbol will be denoted as before by Σ.

It is straightforward to check that for any $f_\sigma \in \mathscr{H}_+\left(f\right)$, conditions $F1 - F4$ hold. We note that all the constants and functions in $F1 - F4$ do not depend on $\sigma \in \Sigma$.

Lemma 4.17. *For any $g_\sigma \in \mathscr{H}_+\left(g\right)$,*

$$\left\|g_\sigma\right\|_{L_\infty\left(\mathbf{R}_+, L_2(\Omega)\right)} \le C_0 = \left\|g\right\|_{L_\infty\left(\mathbf{R}_+, L_2(\Omega)\right)}.$$

Proof. If $g_\sigma\left(t\right) = g\left(t+h\right)$ for some $h \in \mathbf{R}_+$, the statement is obvious. Let us suppose that $g_\sigma \in cl_{L_{2,w}^{loc}\left(\mathbf{R}_+, X\right)}\left\{g\left(t+h\right) : h \ge 0\right\}$. Then there exists a sequence $g^n\left(t\right) = g\left(t+h_n\right)$ converging to g_σ in $L_{2,w}^{loc}\left(\mathbf{R}_+, X\right)$ as $n \to \infty$. Since $\left\|g^n\right\|_{L_\infty\left(\mathbf{R}_+, L_2(\Omega)\right)} \le C_0$, passing to a subsequence $g^n \to \widetilde{g}$ weakly star in $L_\infty\left(\mathbf{R}_+, L_2\left(\Omega\right)\right)$. Hence, $g^n \to \widetilde{g}$ weakly in $L_2\left(\left[0,T\right], L_2\left(\Omega\right)\right)$, $\forall T > 0$, so that $\widetilde{g} = g_\sigma$. Finally,

$$\left\|g_\sigma\right\|_{L_\infty\left(\mathbf{R}_+, L_2(\Omega)\right)} \le \limsup_{n\to\infty} \left\|g^n\right\|_{L_\infty\left(\mathbf{R}_+, L_2(\Omega)\right)} \le C_0.$$

□

Let us now define the family of multivalued maps $F^\sigma : \mathbf{R}_+ \times L_2\left(\Omega\right) \to 2^{L_2(\Omega)}$

$$F^\sigma\left(t,u\right) = \left\{y \in L_2\left(\Omega\right) : y\left(x\right) \in f_\sigma\left(t, u\left(x\right)\right), \text{ a.e. on } \Omega\right\},$$

where $f_\sigma \in \mathscr{H}_+\left(f\right)$.

Let $C_v\left(L_2\left(\Omega\right)\right)$ denote the set of all nonempty, bounded, closed, convex subsets of $L_2\left(\Omega\right)$.

Lemma 4.18. *The following properties hold:*

1. $F^\sigma\left(t,u\right) \in C_v\left(L_2\left(\Omega\right)\right)$, $\forall\left(t,u\right) \in \mathbf{R}_+ \times L_2\left(\Omega\right)$, $\forall \sigma \in \Sigma$.
2. $\mathrm{dist}_H\left(F^\sigma\left(t,u\right), F^\sigma\left(t,v\right)\right) \le C\left\|u-v\right\|_{L_2}$, $\forall u, v \in L_2\left(\Omega\right)$, $\forall t \in \mathbf{R}_+$, $\forall \sigma \in \Sigma$.

Proof. We note that f_σ has compact values and that in view of F1 the set-valued map $f_\sigma\left(t,\cdot\right)$ is continuous in the sense of Hausdorff metric. It is well known that a set-valued map with nonempty compact values is continuous if and only if it is continuous in the sense of Hausdorff metric (see [1]). Hence, $f_\sigma\left(t,\cdot\right)$ is a Carathéodory map and then for any $y(\cdot) \in L_2(\Omega)$, the multivalued map $f_\sigma(t, y(\cdot)) : \Omega \to C_v(\mathbf{R})$ is measurable [2, p.314]. Hence, $f_\sigma(t, y(\cdot))$ has a measurable selection $\xi(\cdot)$, $\xi(x) \in f_\sigma(t, y(x))$ a.e. on Ω [2, p.308]. Using Lemma 4.13, we have $\xi(\cdot) \in L_2(\Omega)$. Consequently, $F^\sigma\left(t,u\right) \in P\left(L_2(\Omega)\right)$. □

Further, using Lemma 4.13 and integrating over Ω, we have

$$\|\xi\|^2_{L_2(\Omega)} \le \int_\Omega (D_1 + D_2 |y(x)|)^2 \, dx \le D < \infty, \forall \xi \in F^\sigma(t, y)$$

Hence, F^σ has bounded values.

Next, we shall prove that F^σ is closed-valued. Let $\{\xi_n\} \in F^\sigma(t, y)$ be a sequence such that $\xi_n \to \xi$ in $L_2(\Omega)$. From $\{\xi_n\}$, we can choose a subsequence (again denoted by ξ_n) converging to ξ almost everywhere on Ω. Since $f_\sigma(t, y(x))$ has closed values, we obtain that $\xi(x) \in f_\sigma(t, y(x))$ a.e. on Ω, that is, $\xi \in F^\sigma(t, y)$.

We must show that F^σ is convex-valued. Let $\xi_1, \xi_2 \in F^\sigma(t, y)$. Using that $f_\sigma(t, y(x))$ is convex-valued, we get

$$(\alpha\xi_1 + (1-\alpha)\xi_2)(x) \in f_\sigma(t, y(x)) \text{ a.e. on } \Omega, \forall \alpha \in [0, 1]$$

Hence, $\alpha\xi_1 + (1-\alpha)\xi_2 \in F^\sigma(t, y), \ \forall \alpha \in [0, 1]$.

Let $u, v \in L_2(\Omega)$ and $\xi \in F^\sigma(t, u)$ be arbitrary. We shall show that

$$dist(\xi, F^\sigma(t, v)) \le C \|u - v\|_{L_2}$$

The maps $\xi(x), \ f_\sigma(t, v(x))$ are measurable. For an arbitrary $\epsilon > 0$, we define the measurable function $\rho_\epsilon(x) = C |u(x) - v(x)| + \epsilon$. The set-valued map $P(x) = B(\xi(x), \rho_\epsilon(x))$, where $B(c, r)$ is a closed ball centered at c with radius r, is measurable [2, p.316]. In view of F1, the intersection $D(x) = P(x) \cap f_\sigma(t, v(x))$ is nonempty for almost all $x \in \Omega$. It follows from [2, p.312] that $D(x)$ is a measurable map. Hence, it has a measurable selection $z_\epsilon(\cdot)$, $z_\epsilon(x) \in D(x)$ a.e. on Ω. We have

$$|\xi(x) - z_\epsilon(x)| \le C |u(x) - v(x)| + \epsilon, \text{ a.e. on } \Omega.$$

Integrating over Ω, we get

$$\|\xi - z_\epsilon\|^2_{L_2} \le C^2 \|u - v\|^2_{L_2} + \epsilon^2\mu(\Omega) + 2\epsilon C \|u - v\|_{L_2} (\mu(\Omega))^{1/2},$$

where $\mu(\Omega)$ is the measure of Ω. As $\epsilon > 0$ is arbitrary, we obtain

$$dist(\xi, F^\sigma(t, v)) \le C \|u - v\|_{L_2}$$

Since $\xi \in F^\sigma(t, u)$ is arbitrary, it follows that $dist(F^\sigma(t, u), F^\sigma(t, v)) \le C \|u - v\|_{L_2}$. In the same way, we have $dist(F^\sigma(t, v), F^\sigma(t, u)) \le \|u - v\|_{L_2}$.

Lemma 4.19. *For any $u \in L_2(\Omega)$, there exists $D(u)$ not depending on σ such that*

$$\sup_{y \in F^\sigma(t,u)} \|y\|_{L_2} \le D(u), \ \forall t \in \mathbf{R}_+.$$

Proof. In view of Lemma 4.13 for any $y \in F^\sigma(t, u)$, we have

$$|y(x)| \leq D_1 + D_2 |u(x)|, \text{ a.e. on } \Omega.$$

After integration, we obtain the required result. □

Lemma 4.20. *There exist* $R_1, R_2 \geq 0$ *such that* $\forall \sigma \in \Sigma$,

$$\operatorname{dist}_H (F^\sigma(t,u), F^\sigma(s,u)) \leq \left(R_1 + R_2 \|u\|_{L_2}\right) \alpha(|t-s|), \forall t, s \in \mathbf{R}_+, \forall u \in L_2(\Omega).$$

Proof. Let $y \in F^\sigma(t,u)$ be arbitrary. The map $x \mapsto f_\sigma(s, u(x))$ is measurable, since $z \mapsto f_\sigma(s,z)$ is continuous (see [2, Theorem 8.2.8.]). For each $\epsilon > 0$, we define the measurable function $x \longmapsto l(|u(x)|)\alpha(|t-s|) + \epsilon = \rho_\epsilon(x)$. Let $B(r,\rho)$ denote a closed ball centered at r with radius ρ. Condition $F2$ implies that the set-valued map $D(x) = B(y(x), \rho_\epsilon(x)) \cap f_\sigma(s, u(x))$ has nonempty values. Moreover, it is measurable and there exists a measurable selection $z_\epsilon(x) \in D(x)$, a.e. $x \in \Omega$ (see Theorems 8.1.3. and 8.2.4. and Corollary 8.2.13. in [2]). Again by $F2$, we get

$$|y(x) - z_\epsilon(x)| \leq (K_1 + K_2 |u(x)|)\alpha(|t-s|) + \epsilon, \text{ a.e. on } \Omega.$$

Hence, after integration, we obtain

$$\begin{aligned}\|y - z_\epsilon\|_{L_2}^2 \leq\ & \epsilon^2 \mu(\Omega) + K_1^2 \alpha^2(|t-s|)\mu(\Omega) + K_2^2 \alpha^2(|t-s|)\|u\|_{L_2}^2 \\ & + 2K_2K_1\alpha^2(|t-s|)\|u\|_{L_2}(\mu(\Omega))^{1/2} + 2\epsilon\alpha(|t-s|)K_1\mu(\Omega) \\ & + 2\epsilon\alpha(|t-s|)K_2\|u\|_{L_2}(\mu(\Omega))^{1/2}.\end{aligned}$$

Passing to the limit as $\epsilon \to 0$, we have

$$\begin{aligned}\operatorname{dist}^2(y, F^\sigma(s,u)) &\leq \alpha^2(|t-s|)\left(K_1^2\mu(\Omega) + K_2^2\|u\|_{L_2}^2 + 2K_2K_1\|u\|_{L_2}(\mu(\Omega))^{1/2}\right) \\ &\leq \alpha^2(|t-s|)\left(K_1^2\mu(\Omega) + K_2^2K_1^2\mu(\Omega) + \left(1 + K_2^2\right)\|u\|_{L_2}^2\right),\end{aligned}$$

and then, there exist $R_1, R_2 > 0$ such that

$$\operatorname{dist}(F^\sigma(t,u), F^\sigma(s,u)) \leq \left(R_1 + R_2\|u\|_{L_2}\right)\alpha(|t-s|).$$

The converse inequality is proved exactly in the same way. □

Corollary 4.1. *For each* $\sigma \in \Sigma$, $u \in L_2(\Omega)$, *the map* $t \mapsto F^\sigma(t,u)$ *is measurable.*

Proof. In view of Lemma 4.20, the map $t \mapsto F^\sigma(t,u)$ is continuous in the Hausdorff metric. This implies that it is lower semicontinuous, that is, for any $t \in \mathbf{R}_+$, $y \in F(t,u)$ and $t_n \to t$, there exist $y_n \in F(t_n, u)$ converging to y. Hence, it is measurable (see [2, Theorem 8.2.1.]). □

Further, we shall define the family of multivalued maps $F_\sigma : \mathbf{R}_+ \times L_2(\Omega) \to 2^{L_2(\Omega)}$

$$F_{\sigma(t)}(u) = F^\sigma(t, u) + g_\sigma(t), \forall t \in \mathbf{R}_+, \forall u \in L_2(\Omega).$$

It follows from the previous results that $\forall t \in \mathbf{R}_+, \forall u \in L_2(\Omega)$, the following properties hold:

(S1) $F_{\sigma(t)}(u) \in C_v(L_2(\Omega))$.
(S2) $\mathrm{dist}_H\left(F_{\sigma(t)}(u), F_{\sigma(t)}(v)\right) \leq C \|u - v\|_{L_2}$.
(S3) For each $u \in L_2(\Omega)$, $t \mapsto F_{\sigma(t)}(u)$ is measurable.
(S4) For any $u \in L_2(\Omega)$, there exists $D(u)$ such that

$$\sup_{y \in F_{\sigma(t)}(u)} \|y\|_{L_2} \leq D(u) + \|g_\sigma(t)\|_{L_2} \leq D(u) + C_0, \text{ a.e. } t \in \mathbf{R}_+.$$

Therefore, conditions $G1 - G3$ hold. Moreover, the functions $k(t) = C$, $n(t) = D(u) + C_0$ do not depend on either σ or (τ, T).

On the other hand, the operator $-A(u) = -\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right)$, with $D(A) = \left\{u \in W_0^{1,p}(\Omega) : A(u) \in L_2(\Omega)\right\}$ is the subdifferential of the proper convex lower semicontinuous function

$$\varphi(u) = \begin{cases} \frac{1}{p} \sum_{i=1}^n \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p dx, & \text{if } u \in W_0^{1,p}(\Omega), \\ +\infty, & \text{otherwise.} \end{cases}$$

Therefore, inclusion (4.15) is a particular case of the abstract one

$$\begin{cases} \frac{du(t)}{dt} + \partial\varphi(u(t)) \in F_{\sigma(t)}(u(t)), \text{ on } (\tau, T), \\ u|_{t=\tau} = u_\tau. \end{cases} \tag{4.16}$$

The operator A is m-dissipative and $\overline{D(A)} = L_2(\Omega)$ (see [33, Sect. 3.2]). Hence, (A) holds and by Proposition 4.6 and Lemma 4.12, we obtain the family of semiprocesses $\{U_\sigma : \sigma \in \Sigma\}$, where Σ is the hull of the symbol $\sigma_0(t) = (f(t, \cdot), g(t))$, which is a compact metric space such that $T(h)\Sigma \subset \Sigma$, $\forall h \in \mathbf{R}_+$.

Let us now prove the properties needed to provide the existence of a uniform global compact attractor. For this purpose, we shall use the fact that if $l(\cdot) \in L_2([\tau, T], X)$, then the integral solution $u(\cdot)$ of the problem

$$\begin{cases} \frac{du(t)}{dt} + \partial\varphi(u(t)) \ni l(t), \text{ on } (\tau, T), \\ u|_{t=\tau} = u_\tau. \end{cases} \tag{4.17}$$

which is unique, is in fact a strong one, that is, $u(\cdot)$ is absolutely continuous on compact sets of (τ, T), a.e. differentiable on (τ, T) and satisfies (4.17) a.e. in (τ, T), and also that the semigroup generated by $-\partial\varphi$ is compact (see [33, Sect. 3.2]).

Proposition 4.7. *For any, $t \in \mathbf{R}_+$ the map $(\sigma, u_0) \mapsto U_\sigma(t, 0, u_0)$ has closed graph.*

Proof. Let $y_n \in U_{\sigma_n}\left(t, 0, u_0^n\right)$ be such that

$$y_n \to y \text{ in } L_2(\Omega),$$

$$u_0^n \to u_0 \text{ in } L_2(\Omega),$$

$$\sigma_n = (f_n, g_n) \to \sigma = (f_\sigma, g_\sigma) \text{ in } C(\mathbf{R}_+, \mathscr{M}) \times L_{2,w}^{loc}(\mathbf{R}_+, L_2(\Omega)).$$

We have to prove that $y \in U_\sigma(t, 0, u_0)$.

There exist sequences $u_n(\cdot) = I\left(u_0^n\right) l_n(\cdot)$, $l_n(s) \in F_{\sigma_n(s)}(u_n(s))$, a.e. in $(0, T)$, such that $y_n = u_n(t)$.

Let $v \in L_2(\Omega)$ be fixed. It follows from $S2$ and $S4$ that for any $y \in L_2(\Omega)$, $\sigma \in \Sigma$,

$$\sup_{\xi \in F_{\sigma(s)}(y)} \|\xi\| \leq D(v) + C\|v\|_{L_2} + C\|y\|_{L_2} + \|g_\sigma(s)\|_{L_2}, \text{ a.e. } s \in (0, T).$$

Therefore, Lemma 4.17 implies that there exist $K_1, K_2 > 0$ for which

$$\|l_n(s)\|_{L_2} \leq \left\|F_{\sigma_n(s)}(u_n(s))\right\|_+ \leq K_1 + K_2\|u_n(s)\|_{L_2}, \text{ a.e. in } (0, T). \tag{4.18}$$

We shall show first the existence of a function $m(\cdot) \in L_\infty(0, T)$, $m(s) \geq 0$, such that $\|l_n(s)\|_{L_2} \leq m(s)$, a.e. in $(0, T)$. Let us introduce the sequence $v_n(\cdot) = I(u_0) l_n(\cdot)$ and let $z(\cdot)$ be the unique solution of

$$\begin{cases} \frac{dz(t)}{dt} + \partial\varphi(z(t)) \ni 0, \text{ on } (0, T), \\ z(0) = u_0. \end{cases}$$

Let $r_0 = \max\{z(s) : s \in [0, T]\}$ and $r_2 = r_1 + r_0$, where $\left\|u_0 - u_0^n\right\|_{L_2} \leq r_1$, $\forall n$. From (4.13), we have

$$\|u_n(s) - z(s)\|_{L_2} \leq \|u_0 - u_0^n\|_{L_2} + \int_0^s \|l_n(r)\|_{L_2} dr$$

and then, by (4.18),

$$\begin{aligned} \|u_n(s)\|_{L_2} &\leq \|z(s)\|_{L_2} + r_1 + \int_0^s \left(K_1 + K_2\|u_n(s)\|_{L_2}\right) dr \\ &\leq r_2 + K_1 s + K_2 \int_0^s \|u_n(s)\|_{L_2} dr. \end{aligned}$$

Hence, by Gronwall's lemma,

$$\|u_n(s)\|_{L_2} \le -\frac{K_1}{K_2} + \left(r_2 + \frac{K_1}{K_2}\right)\exp(K_2 s) = r(s), \forall t \in [0, T].$$

Therefore, by (4.18),

$$\|l_n(s)\|_{L_2} \le K_1 + K_2 r(s) = m(s), \text{ a.e. in } (0, T).$$

The sequence $\{l_n\}$ is then integrable bounded in $L_1(0, T, L_2(\Omega))$, and since the semigroup generated by $-\partial\varphi$ is compact, this implies that the sequence $\{v_n\}$ is precompact in $C([0, T], L_2(\Omega))$ (see [14, Theorem 2.3]). We obtain that there exist subsequences such that

$$v_n \to v \text{ in } C([0, T], L_2(\Omega)),$$
$$l_n \to l \text{ weakly in } L_2(0, T, L_2(\Omega)).$$

Since $l_n \to l$ weakly in $L_1(0, T, L_2(\Omega))$, Lemma 1.3 from [40] implies that $v(\cdot) = I(u_0)\, l(\cdot)$. Using again (4.13), we have $\|u_n(s) - v_n(s)\|_{L_2} \le \left\|u_0^n(s) - u_0(s)\right\|_{L_2}$, $\forall s \in [0, T]$, so that $u_n \to v$ in $C([0, T], L_2(\Omega))$ and $y = v(t)$. To conclude the proof, we have to check that $l(s) \in F_{\sigma(s)}(v(s))$, a.e. on $(0, T)$.

Since $l_n \to l$, $g_n \to g_\sigma$, weakly in $L_2(0, T, L_2(\Omega))$, we have $l_n - g_n = d_n(\cdot) \to l - g_\sigma = d_\sigma(\cdot)$, weakly in $L_2(0, T, L_2(\Omega))$. Then we need to obtain that $d_\sigma(s) \in F^\sigma(s, v(s))$, a.e. on $(0, T)$. Fix $s \in (0, T)$.

Note that since $u_n(s) \to v(s)$ in $L_2(\Omega)$, passing to a subsequence if necessary $u_n(s, x) \to v(s, x)$ for a.a. $x \in \Omega$. Hence, by $F1$,

$$\operatorname{dist}(f_\sigma(s, u_n(s, x)), f_\sigma(s, v(s, x))) \le C\,|u_n(s, x) - v(s, x)| \to 0, \text{ as } n \to +\infty,$$

for a.a. $x \in \Omega$. On the other hand, as $u_n(t, x)$ is bounded, that is $|u_n(s, x)| \le C(x)$, $\forall n$, and f_n converging to f_σ in $C(\mathbf{R}_+, \mathscr{M})$, we get

$$\operatorname{dist}(f_n(s, u_n(s, x)), f_\sigma(s, u_n(s, x))) \to 0, \text{ as } n \to +\infty,$$

for a.a. $x \in \Omega$. Then

$$\operatorname{dist}(d_n(s, x), f_\sigma(s, v(s, x))) \le$$
$$\le \operatorname{dist}(f_\sigma(s, u_n(s, x)), f_\sigma(s, v(s, x))) + \operatorname{dist}(f_n(s, u_n(s, x)), f_\sigma(s, u_n(s, x))) \to 0, \tag{4.19}$$

for a.a. $x \in \Omega$.

In view of [40, Proposition 1.1] for a.a. $s \in (0, T)$

$$d(s) \in \bigcap_{n=1}^{\infty} \overline{co} \bigcup_{k \ge n}^{\infty} d_k(s) = \mathscr{A}(s).$$

Denote $\mathscr{A}_n(s) = co \bigcup_{k \geq n}^{\infty} d_k(s)$. It is easy to see that $z \in \mathscr{A}(s)$ if and only if there exist $z_n \in \mathscr{A}_n(s)$ such that $z_n \to z$, as $n \to \infty$, in $L_2(\Omega)$. Taking a subsequence, we have $z_n(x) \to z(x)$, a.e. in Ω. Since $z_n \in \mathscr{A}_n(s)$, we get

$$z_n(s) = \sum_{i=1}^{N} \lambda_i d_{k_i}(s),$$

where $\lambda_i \in [0, 1]$, $\sum_{i=1}^{N} \lambda_i = 1$ and $k_i \geq n$, $\forall i$. Now, (4.19) implies that for any $\varepsilon > 0$ and a.a. $x \in \Omega$, there exists $n(x, \varepsilon)$ such that

$$d_k(s, x) \subset [a(x) - \varepsilon, b(x) + \varepsilon], \forall k \geq n,$$

where $[a(x), b(x)] = f_\sigma(s, v(s, x))$. Hence,

$$z_n(x) \subset [a(x) - \varepsilon, b(x) + \varepsilon],$$

as well. Passing to the limit, we obtain

$$z(x) \in [a(x), b(x)], \text{ a.e. on } \Omega.$$

Therefore, $z(s) \in F^\sigma(s, v(s))$, so that $d(s) \in \mathscr{A}(s) \subset F^\sigma(s, v(s))$, a.e. on $(0, T)$. It follows that $l(s) \in F_{\sigma(s)}(v(s))$, a.a. on $(0, T)$, as required. □

Corollary 4.2. *For any $t \in \mathbf{R}_+$, the map $(\sigma, u_0) \mapsto U_\sigma(t, 0, u_0)$ has closed values.*

Proposition 4.8. *For any, $t \in \mathbf{R}_+$ the map $(\sigma, u_0) \mapsto U_\sigma(t, 0, u_0)$ is upper semicontinuous, hence w-upper semicontinuous.*

Proof. Suppose that for some (σ, u_0), the map is not upper semicontinuous. Then there exists a neighborhood O of $U_\sigma(t, 0, u_0)$ and sequences $z_n \in U_{\sigma_n}\left(t, 0, u_0^n\right)$, $\sigma_n \to \sigma$ in $C(\mathbf{R}_+, \mathscr{M}) \times L_{2,w}^{loc}(\mathbf{R}_+, L_2(\Omega))$, $u_0^n \to u_0$ in $L_2(\Omega)$, such that $z_n \notin O$. Repeating the same lines of the proof of Proposition 4.7, we can prove that for some subsequence $z_{n_k} \to z \in U_\sigma(t, 0, u_0)$, which is a contradiction. □

Lemma 4.21. *There exists $R_0 > 0$ such that for any bounded set B and $\tau \geq 0$, there is a number $T(B, \tau)$ for which*

$$\|U_\sigma(t, \tau, B)\|_{L_2}^{+} \leq R_0, \forall t \geq T, \forall \sigma \in \Sigma.$$

Proof. Let first $p = 2$. Let $u(\cdot) = I(u_\tau) l(\cdot)$ be an arbitrary integral solution. Multiplying (4.17) by u (note that $\partial\varphi = -\Delta$) and using condition $F4$ and Lemma 4.17, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L_2}^2 + \frac{\epsilon}{2}\|u\|_{L_2}^2 \leq K, \tag{4.20}$$

where $K = M\mu(\Omega) + \frac{1}{2\epsilon}C_0^2$. By Gronwall's lemma, we get

$$\|u(t)\|_{L_2}^2 \leq \exp(-\epsilon(t-\tau))\|u(\tau)\|_{L_2}^2 + \frac{2K}{\epsilon}(1-\exp(-\epsilon(t-\tau))). \quad (4.21)$$

Taking $R_0^2 = \frac{2K}{\epsilon} + \delta$, for some $\delta > 0$, the result follows.

Let now $p > 2$. Since in view of Poincare inequality $(-A(u), u) \geq \gamma \|u\|_{L_p}^p \geq D\|u\|_{L_2}^p$, $\forall u \in D(A)$, where $D > 0$, and as shown before (see (4.18)), there exist $K_1, K_2 > 0$ such that $\forall u \in L_2(\Omega), \sigma \in \Sigma$,

$$\left\| F_{\sigma(t)}(u) \right\|_+ \leq K_1 + K_2 \|u\|_{L_2}, \forall t \in \mathbf{R}_+,$$

we get

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L_2}^2 + D\|u\|_{L_2}^p \leq K_1\|u\|_{L_2} + K_2\|u\|_{L_2}^2.$$

Applying Young's inequality, we obtain that for some $\widetilde{D}, K_3, K_4 > 0$

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L_2}^2 + \frac{\widetilde{D}}{2}\|u\|_{L_2}^2 - K_4 \leq \frac{1}{2}\frac{d}{dt}\|u\|_{L_2}^2 + \frac{D}{2}\|u\|_{L_2}^p \leq K_3, \quad (4.22)$$

and we conclude the proof as before using Gronwall's lemma. □

Let us denote by $B(0, R)$ a closed ball of $L_2(\Omega)$ centered in 0 with radius R.

Lemma 4.22. *For any $R \geq R_0, t, \tau \in \mathbf{R}_+, t \geq \tau, \sigma \in \Sigma$, we have*

$$U_\sigma(t, \tau, B(0, R)) \subset B(0, R).$$

Proof. Let us suppose the opposite, that is, there exist $R \geq R_0, u(\cdot) = I_\sigma(u_\tau)\, l(\cdot)$, $\|u(\tau)\|_{L_2}^2 \leq R^2$ and $t > \tau$ such that $\|u(t)\|_{L_2}^2 > R^2$. Since $u(\cdot)$ is continuous, there exists t_R such that $\|u(t_R)\|_{L_2}^2 = R^2$, $\|u(s)\|_{L_2}^2 > R^2$, $\forall t_R < s \leq t$. Then (4.20) implies

$$\frac{d}{dt}\|u(s)\|_{L_2}^2 \leq -\delta, \forall t_R \leq s \leq t,$$

where $\delta > 0$, and after integration, $\|u(t)\|_{L_2}^2 \leq R^2 - \delta(t - t_R)$, $\forall t_R < s \leq t$, which is a contradiction. □

Corollary 4.3. *For any $B \in \beta(X)$, $\tau \in \mathbf{R}_+$, $\gamma_{0,\Sigma}^\tau(B) \in \beta(X)$.*

For any bounded set B and $\tau, t \in \mathbf{R}_+$, let us introduce the set

$$M(B, \tau, t) = \{l \in L_1(\tau, t; L_2(\Omega)) : u_\sigma(\cdot) = I(u_\tau)\, l(\cdot),$$
$$\times u_\sigma \in \mathscr{D}_{\sigma,\tau}(u_\tau), u_\tau \in B, \sigma \in \Sigma\}.$$

Lemma 4.23. *For any bounded set B and $\tau, t \in \mathbf{R}_+$, the set $M(B, \tau, t)$ is bounded in the space $L_\infty(\tau, t; L_2(\Omega))$.*

Proof. As shown before (see (4.18)), there exist $K_1, K_2 > 0$ such that $\forall u \in L_2(\Omega), \sigma \in \Sigma$,

$$\left\| F_{\sigma(t)}(u) \right\|_+ \le K_1 + K_2 \|u\|_{L_2}, \forall t \in \mathbf{R}_+.$$

Hence, for any $l \in M(B, \tau, t)$,

$$\|l(s)\|_{L_2} \le K_1 + K_2 \|u_\sigma(s)\|_{L_2}, \text{ a.e. } s \in (\tau, t),$$

where $u_\sigma(\cdot) = I(u_\tau) l(\cdot)$. But Lemma 4.22 implies that for any $\sigma \in \Sigma, \tau \le s \le t$, $u_\sigma(s) \in U_\sigma(s, \tau, B)$, $\|u_\sigma(s)\|_{L_2} \le R$ for some $R \ge R_0$, so that the statement follows. □

Proposition 4.9. *There exists a compact set K such that for any $B \in \beta(X)$, $\tau \in \mathbf{R}_+$, there exists $T(B, \tau)$ for which*

$$U_+(t, \tau, B) \subset K, \text{ if } t \ge T.$$

Proof. Set $K = \overline{U_+(1, 0, B(0, R_0))}$. We claim that K is compact. Let

$$y \in U_+(1, 0, B(0, R_0))$$

be arbitrary. Then there exists $u_\sigma(\cdot) = I(u_0) l(\cdot)$, with $\sigma \in \Sigma$, $u_0 \in B(0, R_0)$, such that $y = u_\sigma(1)$. Multiplying the equation

$$\frac{du_\sigma}{dt} - A(u_\sigma) = l \tag{4.23}$$

by u_σ and using Lemma 4.23; the inequality $(-A(u), u) \ge \gamma \|u\|^p_{W^{1,p}}$, $\forall u \in D(A)$, where $\gamma > 0$; and Young's inequality, we have

$$\frac{1}{2}\frac{d}{dt}\|u_\sigma(s)\|^2_{L_2} + \gamma \|u_\sigma(s)\|^p_{W^{1,p}} \le \|l(s)\|_{L_2}\|u_\sigma(s)\|_{L_2} \le \frac{1}{2D}C + \frac{1}{2}D\|u_\sigma(s)\|^p_{L_2}.$$

The continuous injections $W_0^{1,p}(\Omega) \subset L_p(\Omega) \subset L_2(\Omega)$ allow us to choose $D > 0$ such that $D\|u_\sigma(s)\|^p_{L_2} \le \gamma \|u_\sigma(s)\|^p_{W^{1,p}}$. Hence, integrating over $(0, 1)$, we obtain

$$\|u_\sigma(1)\|^2_{L_2} + \gamma \int_0^1 \|u_\sigma(s)\|^p_{W^{1,p}} \le \frac{1}{D}C + \|u_0\|^2_{L_2}. \tag{4.24}$$

Recall that $\varphi(u) = \frac{1}{p}\sum_{i=1}^n \left\| \frac{\partial}{\partial x_i} u \right\|^p_{L_p}$, if $u \in W_0^{1,p}(\Omega)$. Consider first the case where $u_0 \in D(\varphi) = W_0^{1,p}(\Omega)$. In this case, since $l(\cdot) \in L_2(0, 1; L_2(\Omega))$, it is known (see [5, p.189]) that $\varphi(u(t))$ is absolutely continuous in $[0,1]$ and $\frac{d}{ds}\varphi(u(s)) = \left(\partial\varphi(u(s)), \frac{du(s)}{ds}\right)$, a.e. on $(0,1)$. Further, multiplying (4.23) by $s\frac{du_\sigma}{ds}$ we have

$$s\left\|\frac{d}{dt}u_\sigma(s)\right\|_{L_2}^2 + s\frac{d}{ds}\varphi(u(s)) \le s\|l(s)\|_{L_2}\left\|\frac{d}{ds}u_\sigma(s)\right\|_{L_2}$$
$$\le \frac{1}{2}s\|l(s)\|_{L_2}^2 + \frac{1}{2}s\left\|\frac{d}{dt}u_\sigma(s)\right\|_{L_2}^2.$$

Integrating by parts over $(0,1)$, we get

$$\int_0^1 \frac{1}{2}s\left\|\frac{d}{dt}u_\sigma(s)\right\|_{L_2}^2 ds + \varphi(u_\sigma(1)) \le \int_0^1 \varphi(u(s))\,ds + \frac{1}{4}C.$$

Using the fact that the norms $\|u\|_{W^{1,p}}$ and $\left(\sum_{i=1}^n \left\|\frac{\partial}{\partial x_i}u\right\|_{L_p}^p\right)^{\frac{1}{p}}$ are equivalent in $W_0^{1,p}(\Omega)$ and (4.24), we have

$$\varphi(u_\sigma(1)) \le \alpha\left(\frac{1}{D}C + \|u_0\|_{L_2}^2\right) + \frac{1}{4}C. \tag{4.25}$$

Let now consider the general case $u_0 \in L_2(\Omega)$. We take $u_0^n \to u_0$ with $u_0^n \in B(0,R_0)$. From [40, Theorem 3.1], we obtain the existence of a sequence $u_n(\cdot) = I\left(u_0^n\right)l_n(\cdot)$, $l_n(s) \in F_{\sigma(s)}(u_n(s))$, such that $u_n \to u_\sigma$ in $C([0,1], L_2(\Omega))$. Hence, by (4.25) and using the lower semicontinuity of φ, we obtain

$$\varphi(u_\sigma(1)) \le \liminf \varphi(u_n(1)) \le \alpha\left(\frac{1}{D}C + \|u_0\|_{L_2}^2\right) + \frac{1}{4}C.$$

This implies that the set $U_+(1,0,B(0,R_0))$ is bounded in the space $W^{1,p}(\Omega)$. Since the injection $W^{1,p}(\Omega) \subset L_2(\Omega)$ is compact, the set K is compact.

Further, let $B \in \beta(X)$ be arbitrary. Lemma 4.21 implies that for any $\tau \in \mathbf{R}_+$, there exists some $t_1(\tau,B)$ for which $U_+(t,\tau,B) \subset B(0,R_0)$, $\forall t \ge t_1$. Then $\forall\sigma \in \Sigma$, $\forall t = s+1$, where $s \ge t_1(\tau,B)$, we have by Proposition 4.6

$$U_\sigma(t,\tau,B) = U_\sigma(1+s,s,U_\sigma(s,\tau,B))$$
$$= U_{T(s)\sigma}(1,0,U_\sigma(s,\tau,B)) \subset U_{T(s)\sigma}(1,0,B(0,R_0)) \subset K.$$

□

Corollary 4.4. *The family of semiprocesses U_σ is uniformly asymptotically upper semicompact.*

We have proved that the family of semiprocesses generated by (4.15) satisfies all conditions of Theorem 4.7. We can then state the main result of this paper.

Theorem 4.8. *If $F1-F4$ hold for $f : \mathbf{R}_+ \times \mathbf{R} \to C_v(\mathbf{R})$ and $g \in L_\infty(\mathbf{R}_+, L_2(\Omega))$, then the family of semiprocesses U_σ has the uniform global compact attractor Θ_Σ.*

Remark 4.8. If we consider the inclusion

$$\begin{cases} \frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) \in f(t,u) - f_1(u) + g(t), \text{ on } \Omega \times (\tau, T), \\ u\,|_{\partial\Omega} = 0, \\ u\,|_{t=\tau} = u_\tau, \end{cases}$$

where $f_1 : \mathbf{R} \to 2^{\mathbf{R}}$ is a maximal monotone map and $D(f_1) = \mathbf{R}$, then the same result can be proved. We have just to replace in $F4$ the function f by $f - f_1$ and define the operator $-A$ in the following way:

$$-A(u) = \left\{ y \in L_2(\Omega) : y(x) \in -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) + f_1(u(x)), \text{ a.e on } \Omega \right\},$$

$$D(-A) = \left\{ u \in W_0^{1,p}(\Omega) : -A(u) \in L_2(\Omega), \exists \xi \in L_2(\Omega) : \times\, \xi(x) \in f_1(u(x)) \text{ a.e.} \right\},$$

which is the subdifferential of the proper convex lower semicontinuous function

$$\varphi(u) = \begin{cases} \frac{1}{p} \sum_{i=1}^{n} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^p dx + \int_\Omega \int_0^u f_1(s)\, ds, \text{ if } u \in W_0^{1,p}(\Omega), \int_0^u f_1(s)\, ds \in L_1(\Omega), \\ +\infty, \text{ otherwise.} \end{cases}$$

The operator $-\partial\varphi$ generates also a compact semigroup (see [41] or [42]). Small changes are needed in the proofs of Lemma 4.21 and Proposition 4.9 in order to obtain (4.22) and (4.24), respectively. We have to use the existence of $\alpha \in \mathbf{R}, \beta \in L_2(\Omega)$ such that $\forall u \in D(-A), \forall y \in L_2(\Omega), y(x) \in f_1(u(x))$ a.e.,

$$(y, u) \geq \psi(u) - \psi(0) \geq \alpha + (\beta, u),$$

where $\psi(u) = \int_\Omega \int_0^u f_1(s)\, ds$, if $\int_0^u f_1(s)\, ds \in L_1(\Omega)$, and argue as before. The last inequality follows from the fact that $y \in \partial\psi(u)$ and ψ is bounded below by an affine function (see [5]).

Remark 4.9. Let us consider the set $\widetilde{\Sigma} = \mathscr{H}_+(f_0) \times \mathscr{H}_+(g_0)$, that is, the product of the hulls of the functions f_0 and g_0. It is clear that $\Sigma \subset \widetilde{\Sigma}$. It is not difficult to see that all the statements of this section remain valid if we change Σ by $\widetilde{\Sigma}$.

4.3 Applications for Chemical Kinetics Processes and Fields

We can apply this results for classes of diffusion processes from Introduction that can be described by the first-order evolution inclusions (see Chap. 2 and the next chapter).

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Chapter 5
On the Kneser's Property for the Complex Ginzburg–Landau Equation and the Lotka–Volterra System with Diffusion

As we have seen in the previous chapters when we consider the Cauchy problem of a differential equation and uniqueness fails to hold (or it is not known to hold), then we have to consider a set of solutions corresponding to a given initial data. More precisely, let us a consider, for example, an abstract parabolic differential equation

$$\frac{du}{dt} = A\left(t, u\left(t\right)\right), \ \tau \leq t \leq T, \tag{5.1}$$

with initial data

$$u\left(\tau\right) = u_{\tau} \in X, \tag{5.2}$$

where X is a suitable phase space. Assume that we have a theorem of existence of solutions for every $u_0 \in X$, but uniqueness fails, so that more than one solution corresponding to this initial data can be. In such a case, we can consider the set of solutions

$$D_{\tau,T}\left(u_{\tau}\right) = \{u\left(\cdot\right) : u \text{ is a solution of (5.1) such that } u\left(\tau\right) = u_{\tau}\}.$$

We can define then the set of values attained by the solutions at every moment of time $t \geq \tau$, given by

$$K_{\tau,t}\left(u_{\tau}\right) = \{u\left(t\right) : u\left(\cdot\right) \in D_{\tau,T}\left(u_{\tau}\right)\}.$$

This is the attainability set.

Has this set good topological properties? Is it closed? Is it compact? Is it connected?

Let us consider a first-order ordinary differential equation

$$\frac{du}{dt} = f\left(t, u\right),$$

M.Z. Zgurovsky et al., *Evolution Inclusions and Variation Inequalities for Earth Data Processing III*, Advances in Mechanics and Mathematics 27,
DOI 10.1007/978-3-642-28512-7_5, © Springer-Verlag Berlin Heidelberg 2012

where $f : \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}^N$ is continuous. Peano's theorem guarantees the existence of at least one local solution for every initial data $u_\tau \in \mathbf{R}^N$. However, there exist well-known examples where uniqueness fails.

In 1923, Kneser proved the following result (see [8]):

Theorem 5.1. *If all solutions for $u(\tau) = u_\tau$ exist at t, then the attainability set $K_{\tau,t}(u_\tau)$ is closed and connected in $\mathbf{R}^N$.*

In the modern literature, we say that the *Kneser's property* holds if for every moment of time $t \geq \tau$ and every initial data, the attainability set $K_{\tau,t}(u_\tau)$ is compact and connected.

This property has been studied in the last 30 years for several ordinary and partial differential equations. We shall recall some results in this direction.

We shall start with some types of reaction-diffusion equation. In 1993, Kikuchi [15] proved the Kneser's property for the equation

$$\frac{\partial u}{\partial t} - \Delta u = |u|^{\frac{1}{2}}$$

where $x \in \Omega = \mathbf{R}^N$. Also, the case of Ω bounded and open with smooth boundary and Neumann boundary conditions $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$ was considered before (see the references in [15]). This result was generalized in 2001 by Kaminogo [11] to general continuous nonlinearities with sublinear growth. Namely, the authors considered the problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = f(u), \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbf{R}^N$ is open and bounded with smooth boundary and

$$f : \mathbf{R} \to \mathbf{R} \text{ is continuous,}$$
$$|f(u)| \leq C_1 + C_2 |u|.$$

In [1], it is considered the autonomous nonlinear wave equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \Delta u + \beta \frac{\partial u}{\partial t} + f(u) = 0, \\ u|_{\partial\Omega} = 0, \\ u(x,0) = u_0(x), \frac{\partial u}{\partial t}(x,0) = u_1(x), \text{ for } x \in \Omega, \end{cases}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded open set with smooth boundary $\partial\Omega$, $\beta > 0$ and $f : \mathbf{R} \to \mathbf{R}$ is continuous and satisfies the sign condition

$$\lim \inf_{|u| \to \infty} \frac{f(u)}{u} > -\lambda_1,$$

being $\lambda_1 > 0$ the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. When $N \geq 3$, it is assumed that

$$|f(u)| \leq C_0\left(|u|^{\frac{N}{N-2}} + 1\right),$$

which guarantees the existence of at least one globally weak solution for every initial data in $H_0^1(\Omega) \times L_2(\Omega)$. It is proved in [1] that the Kneser's property holds. Weaker conditions are used if $N \leq 2$.

Other important equation of the Mathematical Physics in which uniqueness is not known to hold is the three-dimensional Navier-Stokes system.

Let $\Omega \subset \mathbf{R}^3$ be a bounded open subset with smooth boundary. For given $\nu > 0$, we consider the autonomous Navier-Stokes system

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u = -\nabla p + f, \\ div\, u = 0, \\ u|_{\partial\Omega} = 0,\ u(0,x) = u_0(x), \end{cases} \tag{5.3}$$

where $u(t,x)$ is the velocity of an incompressible fluid, p is the pressure, and f is an external force.

If we consider the usual function spaces

$$\begin{aligned} \mathscr{V} &= \{u \in (C_0^\infty(\Omega))^3 : div\, u = 0\}, \\ H &= cl_{(L_2(\Omega))^3}\mathscr{V},\ V = cl_{(H^1(\Omega))^3}\mathscr{V}, \end{aligned}$$

and assume that $f \in H$, then the following results are well known (see, e.g., [3, 18, 29, 30]):

1. For every $u_0 \in V$, there exists a unique strong solution of problem (5.3) which exists in some interval $[\tau, T(\|u_\tau\|_V))$.
2. For every $u_0 \in H$, there exists at least one weak solution of (5.3) which exists in the whole semiline $[0, +\infty)$.

Therefore, strong solutions are unique but they can be not globally defined in time. On the other hand, weak solutions exist globally in time, but it is not known whether uniqueness holds or not. It is proved in [16, 17] that the set of weak solutions satisfies the Kneser's property with respect to the weak topology of the space H. If we consider the strong topology of H, then this problem remains open, and only some conditional results are obtained.

Other equations for which Kneser's property has been studied are, for example, ordinary differential equations [27], delay differential equations [9–11], reaction-diffusion systems [13, 14, 22], phase-field equations [32], or differential inclusions [25, 31, 32].

In this chapter, we study the Kneser's property, and the asymptotic behavior of solutions as well, for the complex Ginzburg–Landau equation and the Lotka–Volterra system with diffusion. The complex Ginzburg–Landau equation is an important model in many areas of Physics, which appears, for example, in the theory of superconductivity or in chemical turbulence. The Lotka–Volterra system is also a well-known model describing the competition of several species in a common region.

For these equations, it is an open problem so far whether the solution corresponding to a given initial data is unique or not (this is only proved for some values of the parameters). Therefore, if we assume the possibility that more than one solution can exist, it appears the question about the topological properties of the set of values attained by the solutions at a given moment of time. In particular, in this chapter, it is proved that the Kneser's property is true for these systems.

For this aim, we consider first a general reaction-diffusion system

$$\begin{cases} \frac{\partial u^i}{\partial t} - (a\Delta u)_i + f^i(t, u^1, \ldots, u^d) = h^i(t,x),\ i = 1, \ldots, d, \\ u|_{x\in\partial\Omega} = 0,\ u|_{t=\tau} = u_\tau(x), \end{cases} \tag{5.4}$$

where $\Omega \subset \mathbf{R}^N$ is an open bounded set with smooth boundary and with f satisfying suitable growth and dissipative conditions. We observe that we consider the case of nonlinear functions f not having necessarily sublinear growth.

We study the asymptotic behavior of solutions of (5.4), proving the existence of a global compact attractor in both the autonomous and nonautonomous cases. Also, the Kneser's property is proved, and using it, the connectedness of the global attractor is established. As a consequence of these results, the Kneser's property, and the existence of a global compact connected attractor as well, is obtained for the complex Ginzburg–Landau equation and the Lotka–Volterra system with diffusion.

The results of this chapter were proved at first in [13, 14]. The case of unbounded domains was considered in [21, 22]

5.1 Setting of the Problem

Let $\Omega \subset \mathbf{R}^n$ be an open bounded set with smooth boundary $\partial\Omega$.

The nonautonomous complex Ginzburg–Landau equation is the following:

$$\begin{cases} \frac{\partial u}{\partial t} = (1 + \eta i)\,\Delta u + R(t)\,u - (1 + i\beta(t))\,|u|^2\,u + g(t,x), \\ \qquad u\,|_{\partial\Omega} = 0,\ u(x,\tau) = u_\tau(x), \end{cases} \tag{5.5}$$

where $u = u(x,t) = u^1(x,t) + i u^2(x,t)$, $(x,t) \in \Omega \times [\tau, T]$, $g(t) = g^1(t) + g^2(t)\,i \in L_2(\Omega, \mathbf{C})$, $\eta, \beta(t) \in \mathbf{R}$, $R(t) > 0$. We assume that $g^i \in L_2([\tau, T]; L_2(\Omega))$ and also that the functions $R(t)$ and $\beta(t)$ are continuous.

If $n \geq 3$ and we do not assume the condition $|\beta(t)| \leq \sqrt{3}$, then it is not known whether this equation possesses the property of uniqueness of the Cauchy problem or not. Therefore, we have to speak about a set of solutions for a given initial data and, for a particular moment of time $t > \tau$, we can study the properties of the attainability set. More precisely, our aim in this chapter is the following:

1. To prove that the attainability set satisfies the Kneser's property, that is, that it is compact and connected.
2. To check that the global attractor of (5.5) is connected in both the autonomous and nonautonomous cases.

We note that the existence of the global attractor for (5.5) in the phase space $(L_2(\Omega))^2$ was proved in [13] (see also [28]), but the question about the Kneser property and the connectivity of the attractor was left as an open problem. If $n = 1, 2$, in the autonomous case, it is well known that the global attractor exists and that it is connected [29, p.229]. Also, if $n \geq 3$ and $|\beta(t)| \leq \sqrt{3}$, then we have uniqueness of solutions and the existence of the global attractor was established in [4, p.42 and 118] in both the autonomous and nonautonomous cases. Moreover, from the general theory of attractors for semigroups (see, e.g., [29, p.23] or [7]), we obtain that the attractor is connected. In the case $n \geq 3$, $|\beta| > \sqrt{3}$, some results are obtained in the phase space $\left(L_p(\Omega)\right)^2$ with $p > N$, if $(\eta, \beta) \in P(N)$, and $P(N)$ is some subset of $\mathbf{C}$ (see [20]). Other existence and uniqueness results concerning more general cases (including the p-Laplacian and larger classes of nonlinearities) can be found in [23, 24].

In this chapter, it is proved the Kneser's property and the connectedness of the attractor in the phase space $(L_2(\Omega))^2$ for the case where $n \geq 3$ and the condition $\sqrt{\beta} \leq 3$ fails, that is, without any restrictions on the parameters.

We also study the Lotka–Volterra system with diffusion:

$$\begin{cases} \frac{\partial u^1}{\partial t} = D_1 \Delta u^1 + u^1 \left(a_1(t) - u^1 - a_{12}(t) u^2 - a_{13}(t) u^3\right), \\ \frac{\partial u^2}{\partial t} = D_2 \Delta u^2 + u^2 \left(a_2(t) - u^2 - a_{21}(t) u^1 - a_{23}(t) u^3\right), \\ \frac{\partial u^3}{\partial t} = D_3 \Delta u^3 + u^3 \left(a_3(t) - u^3 - a_{31}(t) u^1 - a_{32}(t) u^2\right), \end{cases} \tag{5.6}$$

with Neumann boundary conditions $\frac{\partial u^1}{\partial \nu} |_{\partial\Omega} = \frac{\partial u^2}{\partial \nu} |_{\partial\Omega} = \frac{\partial u^3}{\partial \nu} |_{\partial\Omega} = 0$, where $u^i = u^i(x,t) \geq 0$ and the functions $a_i(t), a_{ij}(t)$ are positive and continuous. Also, D_i are positive constants and $\Omega \subset \mathbf{R}^3$.

Uniqueness of the Cauchy problem for this system has been proved only if we consider solutions confined in an invariant region (e.g., in a parallelepiped $\mathscr{D} = \{\left(u^1, u^2, u^3\right) : 0 \leq u^i \leq k^i\}$ when the parameters do not depend on t) (see [19] and [26]). However, in the general case for initial data just in $(L_2(\Omega))^3$, it is an open problem so far.

Hence, we can consider the same questions as before. Therefore, we prove also for this system the Kneser's property, the existence of a global compact attractor, and its connectivity.

5.2 The Kneser's Property for Reaction-Diffusion Systems

Let $d > 0$ be an integer and $\Omega \subset \mathbf{R}^N$ be a bounded open subset with smooth boundary. We shall denote by $|\cdot|$ the norm in the space $\mathbf{R}^m$, where $m \geq 1$, and by $(\cdot, \cdot)$ the scalar product in $\mathbf{R}^m$. Consider the problem

$$\begin{cases} \frac{\partial u}{\partial t} - a\Delta u + f(t,u) = h(t,x), & (t,x) \in (\tau, T) \times \Omega, \\ u|_{x\in\partial\Omega} = 0, \ u|_{t=\tau} = u_\tau(x), \end{cases} \tag{5.7}$$

where $\tau, T \in \mathbf{R}$, $T > \tau$, $x \in \Omega$, $u = \left(u^1(t,x), \ldots, u^d(t,x)\right)$, $f = \left(f^1, \ldots, f^d\right)$, a is a real $d \times d$ matrix with a positive symmetric part $\frac{a+a^t}{2} \geq \beta I$, $\beta > 0$, $h \in L_2(\tau, T; (L_2(\Omega))^d)$. Moreover, $f = \left(f^1(t,u), \ldots, f^d(t,u)\right)$ is jointly continuous on $[\tau, T] \times \mathbf{R}^d$ and satisfies the following conditions:

$$\sum_{i=1}^{d} |f^i(t,u)|^{\frac{p_i}{p_i-1}} \leq C_1(1 + \sum_{i=1}^{d} |u^i|^{p_i}), \tag{5.8}$$

$$(f(t,u), u) \geq \alpha \sum_{i=1}^{d} |u^i|^{p_i} - C_2, \tag{5.9}$$

where $p_i \geq 2, \alpha, C_1, C_2 > 0$.

We shall use the following standard notation: $H = (L_2(\Omega))^d$, $V = \left(H_0^1(\Omega)\right)^d$, V' is the dual space of V. By $\|\cdot\|$, $\|\cdot\|_V$, we denote the norm in H and V, respectively. For $p = (p_1, \ldots, p_d)$, we define the spaces

$$L_p(\Omega) = L_{p_1}(\Omega) \times \cdots \times L_{p_d}(\Omega),$$
$$L_p\left(\tau, T; L_p(\Omega)\right) = L_{p_1}\left(\tau, T; L_{p_1}(\Omega)\right) \times \cdots \times L_{p_d}\left(\tau, T; L_{p_d}(\Omega)\right).$$

We set $q = (q_1, \ldots, q_d)$, where $\frac{1}{p_i} + \frac{1}{q_i} = 1$.

We say that the function $u(\cdot)$ is a weak solution of (5.7) if $u \in W_{\tau,T} = L_p(\tau, T; L_p(\Omega)) \cap L_2(\tau, T; V) \cap C([\tau, T]; H)$, $\frac{du}{dt} \in L_2(\tau, T; V') + L_q(\tau, T; L_q(\Omega))$, $u(\tau) = u_\tau$, and

$$\int_\tau^T \left\langle \frac{du}{dt}, \xi \right\rangle dt + \int_\tau^T \int_\Omega (\nabla(au), \nabla\xi)\, dxdt$$
$$+ \int_\tau^T \int_\Omega (f(x,t,u), \xi)\, dxdt = \int_\tau^T \int_\Omega (h, \xi)\, dxdt, \tag{5.10}$$

for all $\xi \in L_p(\tau, T; L_p(\Omega)) \cap L_2(\tau, T; V)$, where $\langle \cdot,\cdot \rangle$ denotes pairing in the space $V' + L_q(\Omega)$ and $(\nabla u, \nabla v) = \sum_{i=1}^d \left(\nabla u^i, \nabla v^i\right)$.

Under conditions (5.8)–(5.9), it can be proved in a standard way that for any $u_\tau \in H$, there exists at least one weak solution $u = u(t,x)$ of (5.7). This result is obtained by using the Galerkin method with the special basis of the eigenvalues of the operator $-\Delta$ in $H_0^1(\Omega)$. For the details, see [4, p.284] (or [12, Theorem 3.9], where the scalar case is considered).

It follows also that any weak solution satisfies $\frac{du}{dt} \in L_q(\tau, T; H^{-r}(\Omega))$, where $r = (r_1, \ldots, r_d)$, $r_i = \max\{1; N(1/q_i - 1/2)\}$, and

$$L_q(0,T; H^{-r}(\Omega)) = L_{q_1}(0,T; H^{-r_1}(\Omega)) \times \cdots \times L_{q_d}(0,T; H^{-r_d}(\Omega)).$$

It is well known [4, p.285] that the function $t \mapsto \|u(t)\|^2$ is absolutely continuous on $[\tau, T]$ and $\frac{d}{dt}\|u(t)\|^2 = 2\left\langle \frac{du}{dt}, u\right\rangle$ for a.a. $t \in (\tau, T)$. Hence, it is standard to prove using (5.8)–(5.9) and the properties of the matrix a that every weak solution u of (5.7) from the class $W_{\tau,T}$ satisfies for all $t \geq s$, $t, s \in [\tau, T]$, the following estimate:

$$\|u(t)\|^2 + 2\beta \int_s^t \|u(\tau)\|_V^2 d\tau + \alpha \sum_{i=1}^d \int_s^t \|u^i(r)\|_{L_{p_i}(\Omega)}^{p_i} dr \leq \|u(s)\|^2 + C \int_s^t (\|h(r)\|^2 + 1) dr. \tag{5.11}$$

Indeed, multiplying the equation in (5.7) by u and using (5.9) and

$$(a\nabla u, \nabla u) = \left(\frac{a + a^t}{2}\nabla u, \nabla u\right) \geq \beta \|u\|_V^2,$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \beta\|u\|_V^2 + \alpha \sum_{i=1}^d \left\|u^i(t)\right\|_{L_{p_i}(\Omega)}^{p_i} \leq \frac{1}{2}\|h(t)\|^2 + \frac{1}{2}\|u\|^2 + C_2$$
$$\leq \frac{1}{2}\|h(t)\|^2 + C_3 \sum_{i=1}^d \left\|u^i\right\|_{L_{p_i}(\Omega)}^2 + C_2 \leq \frac{1}{2}\|h(t)\|^2 + \frac{\alpha}{2} \sum_{i=1}^d \left\|u^i\right\|_{L_{p_i}(\Omega)}^{p_i} + C_4. \tag{5.12}$$

After integration, (5.11) follows.

We shall prove further a compactness property of solutions. For this aim, we will need the following classical results [18].

Theorem 5.2. *(Compactness Theorem) Let $X \subset H \subset Y$ be Banach spaces, where X, Y are reflexive and the embeddings are continuous. Moreover, the embedding*

$X \subset H$ is compact. Let

$$W = \{u : u \in L_{p_0}(\tau, T; X), \frac{du}{dt} \in L_{p_1}(\tau, T; Y)\},$$

where $1 < p_i < \infty$, $i = 0, 1$. Then the Banach space W, endowed with the norm $\|u\|_{L_{p_0}(\tau,T;X)} + \left\|\frac{du}{dt}\right\|_{L_{p_1}(\tau,T;Y)}$, is compactly embedded in $L_{p_0}(\tau, T; H)$.

Lemma 5.1. *Let $\Omega \subset \mathbf{R}^N$ be a bounded subset and let $g_n(t), g(t)$ be functions from $L_q(\Omega)$, $1 < q < \infty$, such that*

$$\|g_n\|_{L_q(\Omega)} \le C, \ \forall n,$$
$$g_n(x) \to g(x) \text{ for a.a. } x \in \Omega.$$

Then

$$g_n \to g \text{ weakly in } L_q(\Omega).$$

Now, we are ready to prove our result.

Lemma 5.2. *Let conditions (5.8)–(5.9) hold and let $\{u_n\} \subset W_{\tau,T}$ be an arbitrary sequence of solutions of (5.7) with $u_n(\tau) \to u_\tau$ weakly in H. Then there exists a subsequence such that $u_n(t_n) \to u(t_0)$ in H, for any $t_n \to t_0$, $t_n, t_0 \in (\tau, T]$, where $u(\cdot) \in W_{\tau,T}$ is a weak solution of (5.7) and $u(\tau) = u_\tau$.*

Proof. In view of inequality (5.11), the sequence $\{u_n(\cdot)\}$ is bounded in $W_{\tau,T}$, and $f(t, u_n)$ is bounded in $L_q(\tau, T; L_q(\Omega))$. It follows also that $\{\frac{\partial u_n}{\partial t}(\cdot)\}$ is bounded in $L_q(\tau, T; H^{-r}(\Omega))$. Hence, up to a subsequence, u_n converges to a function u in the following sense:

$$u_n \to u \text{ weakly star in } L_\infty(\tau, T; H), \tag{5.13}$$

$$u_n \to u \text{ weakly in } L_p(\tau, T; L_p(\Omega)), \tag{5.14}$$

$$u_n \to u \text{ weakly in } L_2(\tau, T; V), \tag{5.15}$$

$$\frac{du_n}{dt} \to \frac{du}{dt} \text{ weakly in } L_q(\tau, T; H^{-r}(\Omega)). \tag{5.16}$$

By the Compactness Theorem 5.2, we have that

$$u_n \to u \text{ in } L_2(\tau, T; H), \tag{5.17}$$

$$u_n(t) \to u(t) \text{ in } H \text{ for a.a. } t \in (\tau, T), \tag{5.18}$$

$$u_n(t, x) \to u(t, x) \text{ for a.a. } (t, x) \in (\tau, T) \times \Omega. \tag{5.19}$$

On the other hand, for any sequence $t_n \to t_0$, $t_n, t_0 \in [\tau, T]$, it follows that

$$u_n(t_n) \to u(t_0) \text{ weakly in } H. \tag{5.20}$$

Indeed, as $\frac{\partial u_n}{\partial t}$ is a bounded sequence of the space $L_q(0,T;H^{-r}(\Omega))$, we have that $u_n(t) : [0,T] \to H^{-r}(\Omega)$ is an equicontinuous family of functions. By (5.11) for each fixed $t \in [0,T]$, the sequence $u_n(t)$ is bounded in H, so that the compact embedding $L_2(\Omega) \subset H^{-r_i}(\Omega_n)$, $\forall i$, implies that it is precompact in $H^{-r}(\Omega)$. Applying the Ascoli–Arzelà theorem, we deduce that $u_n(t)$ is a precompact sequence in $C([\tau,T], H^{-r}(\Omega))$. Hence, since $u_n \to u$ weakly in $L_2(0,T;H^{-r}(\Omega))$, we have $u_n \to u$ in $C([\tau,T], H^{-r}(\Omega))$. The boundedness of $u_n(t_n)$ in H implies then by a standard argument that $u_n(t_n) \to u(t_0)$ weakly in H.

In particular, $u_n(\tau) \to u(\tau)$ weakly in H, so that $u(\tau) = u_\tau$.

Let us prove that u is a solution of (5.7). We note that (5.19) implies $f(t,u_n(t,x)) \to f(t,u(t,x))$ for a.a. $(t,x) \in (\tau,T) \times \Omega$, and then by the boundedness of $f(t,u_n)$ in $L_q(\tau,T;L_q(\Omega))$ and Lemma 5.1, we have that $f(\cdot,u_n(\cdot))$ converges to $f(\cdot,u(\cdot))$ weakly in $L_q(\tau,T;L_q(\Omega))$, and then passing to the limit in the inequality

$$\int_\tau^T \left\langle \frac{du_n}{dt}, \xi \right\rangle dt + \int_\tau^T \int_\Omega (a\nabla u_n, \nabla \xi)\, dxdt + \int_\tau^T \int_\Omega (f(t,u_n), \xi)\, dxdt$$
$$= \int_\tau^T \int_\Omega (h,\xi)\, dxdt,$$

for any $\xi \in L_2(\tau,T;V) \cap L_p\left(\tau,T;L_p(\Omega)\right)$, we obtain that $u(\cdot) \in W_{\tau,T}$ is a solution of (5.7) such that $u(\tau) = u_\tau$.

Let now $t_n \to t_0, t_n, t_0 \in (\tau,T]$. We know by (5.20) that $u_n(t_n) \to u(t_0)$ weakly in H. Hence, $\liminf \|u_n(t_n)\| \geq \|u(t_0)\|$. If we prove that

$$\limsup \|u_n(t_n)\| \leq \|u(t_0)\|,$$

then $u_n(t_n) \to u(t_0)$ in H, as desired.

In a similar way as in the proof of inequality (5.11), one can see that u_n and u satisfy the following inequalities,

$$\begin{aligned} \|u_n(t)\|^2 &\leq \|u_n(s)\|^2 + K(t-s) + 2\int_s^t (h(v), u_n(v))dv, \\ \|u(t)\|^2 &\leq \|u(s)\|^2 + K(t-s) + 2\int_s^t (h(v), u(v))dv, \end{aligned} \tag{5.21}$$

for all $t \geq s$, $t,s \in [\tau,T]$, where the constant $K > 0$ does not depend on n. Therefore, the functions $J_n(t) = \|u_n(t)\|^2 - Kt - 2\int_\tau^t (h(v), u_n(v))dv$, $J(t) = \|u(t)\|^2 - Kt - 2\int_\tau^t (h(v), u_n(v))dv$ are continuous and nonincreasing on $[\tau,T]$.

Moreover, since $u_n(t) \to u(t)$ in H for a.a. $t \in (\tau, T)$, $u_n \to u$ in $L_2(\tau, T; H)$, we have $J_n(t) \to J(t)$ for a.a. $t \in (\tau, T)$.

We state that $\limsup J_n(t_n) \leq J(t_0)$. Indeed, let $\tau < t_m < t_0$ be such that $J_n(t_m) \to J(t_m)$. We can assume that $t_m < t_n$. Since J_n are nonincreasing, we obtain

$$J_n(t_n) - J(t_0) \leq |J_n(t_m) - J(t_m)| + |J(t_m) - J(t_0)|.$$

For any $\varepsilon > 0$, there exist $t_m(\varepsilon)$ and $n_0(t_m)$ such that $J_n(t_n) - J(t_0) \leq \varepsilon$, for all $n \geq n_0$, and the result follows.

Hence, since $\int_\tau^{t_n}(h(v), u_n(v))dv \to \int_\tau^t (h(v), u(v))dv$, we have

$$\limsup J_n(t_n) = \limsup \|u_n(t_n)\|^2 - K_2 t - \int_\tau^t (h(v), u(v))dv$$

$$\leq \|u(t_0)\|^2 - K_2 t - \int_\tau^t (h(v), u(v))dv.$$

Therefore, $\limsup \|u_n(t_n)\| \leq \|u(t_0)\|$. □

It is important to point out here that under conditions (5.8)–(5.9), in general, more than one solution can exist (at the end of this section an example is given in the case where the function f can depend on $x \in \Omega$). Therefore, the following question appears naturally: is the set of values attained by the solution at any moment of time t connected? We shall give a positive answer to this question.

We define a sequence of smooth functions $\psi_k : \mathbf{R}_+ \to [0,1]$ satisfying

$$\psi_k(s) = \begin{cases} 1, \text{ if } 0 \leq s \leq k, \\ 0 \leq \psi_k(s) \leq 1, \text{ if } k \leq s \leq k+1, \\ 0, \text{ if } s \geq k+1. \end{cases} \tag{5.22}$$

For every $k \geq 1$, we put $f_k^i(t,u) = \psi_k(|u|) f^i(t,u) + (1 - \psi_k(|u|)) g^i(u)$, where $g^i(u) = \left|u^i\right|^{p_i - 2} u^i$. Then $f_k \in \mathbf{C}([\tau, T] \times \mathbf{R}^d; \mathbf{R}^d)$ and $\sup_{t \in [\tau,T]} \sup_{|u| \leq A} |f_k(t,u) - f(t,u)| \to 0$, *as* $k \to \infty$, for any $A > 0$. Let $\rho_\varepsilon : \mathbf{R}^d \to \mathbf{R}_+$ be a mollifier, that is, $\rho_\epsilon \in \mathbf{C}_0^\infty\left(\mathbf{R}^d; \mathbf{R}\right)$, $supp\ \rho_\epsilon \subset B_\epsilon$, $\int_{\mathbf{R}^d} \rho_\epsilon(s)\,ds = 1$ and $\rho_\epsilon(s) \geq 0$ for all $s \in \mathbf{R}^d$, where $B_\epsilon = \left\{u \in \mathbf{R}^d : |u| \leq \epsilon\right\}$. We define the functions $f_k^\epsilon(t,u) = \int_{\mathbf{R}^d} \rho_\epsilon(s) f_k(t, u-s)ds$. Since for any $k \geq 1$ f_k is uniformly continuous on $[\tau, T] \times B_{k+1}$, there exist $\epsilon_k \in (0,1)$ such that for all u satisfying $|u| \leq k$, and for all s for which $|u - s| < \epsilon_k$, we have

$$\sup_{t \in [\tau,T]} |f_k(t,u) - f_k(t,s)| \leq \frac{1}{k}. \tag{5.23}$$

We put $f^k(t,u) = f_k^{\epsilon_k}(t,u)$. Then $f^k(t,\cdot) \in \mathbf{C}^{\infty}(\mathbf{R}^d;\mathbf{R}^d)$, for all $t \in [\tau,T]$, $k \geq 1$.

For further arguments, we need the following technical result:

Lemma 5.3. *For all $k \geq 1$, the following statements hold:*

$$\sup_{t\in[\tau,T]}\sup_{|u|\leq A} |f^k(t,u) - f(t,u)| \to 0, \text{ as } k \to \infty, \forall\, A > 0, \tag{5.24}$$

$$\sum_{i=1}^{d} |f^{ki}(t,u)|^{\frac{p_i}{p_i-1}} \leq D_1(1 + \sum_{i=1}^{d} |u^i|^{p_i}),\ \left(f^k(t,u),u\right) \geq \eta \sum_{i=1}^{d} |u^i|^{p_i} - D_2, \tag{5.25}$$

$$\left(f_u^k(t,u)w,w\right) \geq -D_3(k)\,|w|^2,\ \forall u,w, \tag{5.26}$$

where $D_3(k)$ is a nonnegative number, and the positive constants D_1, $D_2 \geq C_2$, η do not depend on k. Here, f_u^k denotes the jacobian matrix of f^k with respect to u.

Proof. Since in view of (5.23) for any $t \in [\tau,T]$ and any u such that $|u| \leq k$ we have

$$|f^k(t,u) - f_k(t,u)| \leq \int_{\mathbf{R}^d} \rho_{\epsilon_k}(u-s)|f_k(t,s) - f_k(t,u)|ds \leq \frac{1}{k},$$

we obtain that for any $A > 0$ and any u such that $|u| \leq A$, we get

$$|f^k(t,u) - f(t,u)| \leq |f^k(t,u) - f_k(t,u)| + |f_k(t,u) - f(t,u)| \leq \frac{1}{k}, \forall\, k \geq A.$$

Hence, (5.24) holds. Let us verify estimates (5.8)–(5.9) for the function f_k. It is clear that

$$(f_k(t,u),u) = \psi_k\left(|u|\right)\left(f\left(t,u\right),u\right) + \left(1 - \psi_k\left(|u|\right)\right)\left(g\left(u\right),u\right)$$

$$\geq \psi_k\left(|u|\right)\left(\alpha \sum_{i=1}^{d} |u^i|^{p_i} - C_2\right) + \left(1 - \psi_k\left(|u|\right)\right) \sum_{i=1}^{d} |u^i|^{p_i} \geq \widetilde{\alpha} \sum_{i=1}^{d} |u^i|^{p_i} - C_2,$$

where $\widetilde{\alpha} = \min\{1,\alpha\}$. Also,

$$\sum_{i=1}^{d} |f_k^i(t,u)|^{\frac{p_i}{p_i-1}} \leq D\left(\sum_{i=1}^{d} |f^i(t,u)|^{\frac{p_i}{p_i-1}} + \sum_{i=1}^{d} \left|g^i\left(u\right)\right|^{\frac{p_i}{p_i-1}}\right)$$

$$\leq D\left(C_1(1 + \sum_{i=1}^{d} |u^i|^{p_i}) + \sum_{i=1}^{d} |u^i|^{p_i}\right) \leq D\left(C_1 + 1\right)\left(\sum_{i=1}^{d} |u^i|^{p_i} + 1\right),$$

for some constant $D > 0$. Thus, for $\widetilde{C}_1 = D(C_1 + 1)$, $\widetilde{C}_2 = C_2$, $\widetilde{\alpha} = \min\{1, \alpha\}$, we have (5.8)–(5.9) for the function f_k.

Now, let us consider f^k. Using (5.8) for f_k, we have

$$\sum_{i=1}^{d} |f^{ki}(t,u)|^{\frac{p_i}{p_i-1}} \leq \sum_{i=1}^{d} \left(\left(\int_{\mathbf{R}^d} \rho_{\epsilon_k}(s) ds \right)^{\frac{1}{p_i-1}} \int_{\mathbf{R}^d} \rho_{\epsilon_k}(s) \left| f^i(t,u-s) \right|^{\frac{p_i}{p_i-1}} ds \right)$$

$$\leq C \sum_{i=1}^{d} \int_{\mathbf{R}^d} \rho_{\epsilon_k}(s)(1 + |u^i - s^i|^{p_i}) ds \leq$$

$$\leq \widetilde{C} \sum_{i=1}^{d} \int_{\mathbf{R}^d} \rho_{\epsilon_k}(s)(1 + |u^i|^{p_i} + \epsilon_k^{p_i}) ds \leq D_1(1 + \sum_{i=1}^{d} |u^i|^{p_i}).$$

Further, by using the Young's inequality and, again, estimates (5.8)–(5.9) for the function f_k, we obtain

$$\left(f^k(t,u), u\right) = \int_{\mathbf{R}^d} \rho_{\epsilon_k}(s)(f_k(t,u-s), u-s) ds + \int_{\mathbf{R}^d} \rho_{\epsilon_k}(s) \left(f_k(t,u-s), s\right) ds$$

$$\geq \int_{\mathbf{R}^d} \rho_{\epsilon_k}(s)(\widetilde{\alpha} \sum_{i=1}^{d} \left|u^i - s^i\right|^{p_i} - C_2) ds$$

$$- \int_{\mathbf{R}^d} \rho_{\epsilon_k}(s) \sum_{i=1}^{d} \left(\frac{\widetilde{\alpha}}{2\widetilde{C}_1} \left| f_k^i(t,u-s) \right|^{\frac{p_i}{p_i-1}} + K_0 \left|s^i\right|^{p_i} \right) ds$$

$$\geq \int_{\mathbf{R}^d} \rho_{\epsilon_k}(s)(\widetilde{\alpha} \sum_{i=1}^{d} \left|u^i - s^i\right|^{p_i} - C_2) ds$$

$$- \int_{\mathbf{R}^d} \rho_{\epsilon_k}(s) \left(\frac{\widetilde{\alpha}}{2} \sum_{i=1}^{d} |u^i - s^i|^{p_i} + K_0 \sum_{i=1}^{d} (\epsilon_k)^{p_i} + \frac{\widetilde{\alpha}}{2} \right) ds$$

$$\geq \int_{\mathbf{R}^d} \rho_{\epsilon_k}(s)(\frac{\widetilde{\alpha}}{2} \sum_{i=1}^{d} \left|u^i - s^i\right|^{p_i} - C_2) ds - K_1 \geq \eta \sum_{i=1}^{d} \left|u^i\right|^{p_i} - D_2,$$

for some constant $K > 0$, where in the last inequality, we have used that for some $D > 0$,

$$\left|u^i\right|^{p_i} = \left|u^i - s^i + s^i\right|^{p_i} \leq D\left(\left|u^i - s^i\right|^{p_i} + \left|s^i\right|^{p_i}\right) \leq D\left(\left|u^i - s^i\right|^{p_i} + \epsilon_k^{p_i}\right).$$

Let us show that for $|u| > k + 2$, we have $\left(f_u^k(t,u)w, w\right) \geq 0$, for all $w \in \mathbf{R}^d$. Indeed, if $|u| > k + 1$, we have

$$\frac{\partial f_k^i(t,u)}{\partial u_i} = \frac{\partial g^i(t,u)}{\partial u_i} = (p_i - 1)\left|u^i\right|^{p_i-2} \geq 0, \ \frac{\partial f_k^i(t,u)}{\partial u_j} = 0, \text{if } j \neq i,$$

so the matrix $f_{ku}(t,u)$ satisfies $(f_{ku}(t,u)w,w) \geq 0$. Then for $|u| > k+2$, we get

$$\left(f_u^k(t,u)w,w\right) = \int_{\mathbf{R}^d} \rho_{\epsilon_k}(s)\left(f_{ku}(t,u-s)w,w\right)ds \geq 0.$$

Finally, if $|u| \leq k+2$, we have

$$|\left(f_u^k(t,u)w,w\right)| \leq |w|^2 \int_{\mathbf{R}^d} |\nabla \rho_{\epsilon_k}(u-s)||f_k(t,s)|ds \leq D_3(k)|w|^2.$$

□

For arbitrary $u_\tau \in H$ and $T > \tau$, we define the following set:

$$D_{\tau,T}(u_\tau) = \{u(\cdot) : u(\cdot) \in W_{\tau,T} \text{ is a weak solution of (5.7)}, u(\tau) = u_\tau\},$$

and for any $t \in [\tau, T]$ its corresponding attainability set

$$K_t(u_\tau) = \{u(t) : u(\cdot) \in D_{\tau,T}(u_\tau)\}.$$

Our aim is to prove the connectedness of the set $K_t(u_\tau) \subset H$ for any $t \in [\tau, T]$. We note that from Lemma 5.2, we immediately obtain the compactness of $K_t(u_\tau)$ in H.

Theorem 5.3. *The set $K_t(u_\tau)$ is connected in H for any $t \in [\tau, T]$.*

Proof. The case $t = \tau$ is obvious. Suppose then that for some $t^* \in (\tau, T]$ the set $K_{t^*}(u_\tau)$ is not connected. Then there exist two compact sets $A_1, A_2 \subset H$ such that $A_1 \cup A_2 = K_{t^*}(u_\tau)$, $A_1 \cap A_2 = \emptyset$. Let $u_1(\cdot),\ u_2(\cdot) \in D_{\tau,T}(u_\tau)$ be such that $u_1(t^*) \in U_1$, $u_2(t^*) \in U_2$, where U_1, U_2 are disjoint open neighborhoods of A_1, A_2 respectively.

Let $u_i^k(t,\gamma)$, $i = 1,2$, be equal to $u_i(t)$, if $t \in [\tau, \gamma]$, and let $u_i^k(t,\gamma)$ be a solution of the problem

$$\begin{cases} \frac{\partial u}{\partial t} - a\Delta u + f^k(t,u) = h(t,x), \quad (t,x) \in (\gamma, T) \times \Omega, \\ u|_{x\in\partial\Omega} = 0, \ u|_{t=\gamma} = u_i(\gamma, x), \end{cases} \tag{5.27}$$

if $t \in [\gamma, T]$. In view of Lemma 5.3 for all $k \geq 1$, the function f^k satisfies the same type of conditions as f, so that problem (5.27) has at least one weak solution from $W_{\gamma,T}$. It follows from (5.26) that this solution is unique. Indeed, let $w = v - u$, where $v, u \in W_{\gamma,T}$ are solutions of (5.27). Then (5.26) and the properties of the matrix a imply

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 + \beta\|\nabla w\|^2 \leq D_3(k)\|w\|^2, \text{ for a.a. } t \in (\gamma, T).$$

Hence, from the Gronwall's lemma, we obtain

$$\|w(t)\| \leq \|w(\gamma)\| \exp\left(2D_3(k)\,(t-\gamma)\right), \tag{5.28}$$

and then we have $w(t) \equiv 0$ on $[\gamma, T]$. Arguing as in the proof of (5.11) and using (5.25), one can easily obtain

$$\|u_i^k(t)\|^2 + 2\beta \int_\gamma^t \|u_i^k(\tau)\|_V^2 d\tau + \eta \sum_{i=1}^d \int_\gamma^t \|u_i^k(\tau)\|_{L_{p_i}(\Omega)}^{p_i} d\tau$$

$$\leq \|u_i(\gamma)\|^2 + K_1 \int_\gamma^t (\|h(v)\|^2 + 1) dv. \tag{5.29}$$

Lemma 5.4. *The maps $\gamma \mapsto u_i^k(t,\gamma)$ are continuous for each fixed $k \geq 1$ and $t \in [\tau, T]$.*

Proof. We shall omit the index i for simplicity of notation.

Let $\gamma \to \gamma_0$. Consider first the case where $\gamma > \gamma_0$, that is, $\gamma \searrow \gamma_0$. If $t \leq \gamma_0 < \gamma$, then $u^k(t,\gamma) = u(t) = u^k(t,\gamma_0)$. We note also that $u^k(t,\gamma) = u(t)$, for all $t \in [\tau,\gamma]$. Now, if $t > \gamma_0$, then we can assume that $t > \gamma$, so that $u^k(t,\gamma)$ is the solution of (5.27) on $[\gamma, T]$ such that $u^k(\gamma,\gamma) = u(\gamma)$ and $u^k(t,\gamma_0)$ is the solution of (5.27) on $[\gamma_0, T]$ such that $u^k(\gamma_0,\gamma_0) = u(\gamma_0)$. Further, $u(\gamma) \to u(\gamma_0)$, $u^k(\gamma,\gamma_0) \to u(\gamma_0)$, as $\gamma \to \gamma_0$, by continuity. Using (5.28) for $w(t) = u^k(t,\gamma) - u^k(t,\gamma_0)$, we have

$$\|u^k(t,\gamma) - u^k(t,\gamma_0)\| \leq \|u^k(\gamma,\gamma) - u^k(\gamma,\gamma_0)\| \exp\left(2D_3(k)\,(t-\gamma)\right)$$

$$= \|u(\gamma) - u^k(\gamma,\gamma_0)\| \exp\left(2D_3(k)\,(t-\gamma)\right)$$

$$\leq (\|u(\gamma) - u(\gamma_0)\| + \|u(\gamma_0) - u^k(\gamma,\gamma_0)\|) \exp\left(2D_3(k)\,(t-\gamma)\right) \to 0, \text{ as } \gamma \to \gamma_0.$$

Let now $\gamma < \gamma_0$, that is, $\gamma \nearrow \gamma_0$. If $t < \gamma_0$, then we can assume that $t < \gamma$, so that $u^k(t,\gamma) = u(t) = u^k(t,\gamma_0)$. We note also that $u^k(t,\gamma_0) = u(t)$, for all $t \in [\tau,\gamma_0]$. If $t \geq \gamma_0 > \gamma$, then $u^k(t,\gamma)$ is the solution of (6.13) on $[\gamma, T]$, $u^k(\gamma,\gamma) = u(\gamma)$ and $u^k(t,\gamma_0)$ is the solution of (5.27) on $[\gamma_0, T]$ such that $u^k(\gamma_0,\gamma_0) = u(\gamma_0)$. Hence,

$$\|u^k(t,\gamma) - u^k(t,\gamma_0)\| \leq \|u^k(\gamma_0,\gamma) - u^k(\gamma_0,\gamma_0)\| \exp\left(2D_3(k)\,(t-\gamma_0)\right)$$

$$= \|u^k(\gamma_0,\gamma) - u(\gamma_0)\| \exp\left(2D_3(k)\,(t-\gamma_0)\right).$$

To finish the proof of the continuity, we have to check that $\|u^k(\gamma_0,\gamma) - u(\gamma_0)\| \to 0$, as $\gamma \nearrow \gamma_0$.

Since $u(\cdot) \in C([0,T]; H)$ from (5.29), we can find a constant $R > 0$, which does not depend neither on γ nor k, such that

$$\begin{aligned} &\|u^k(t,\gamma)\| \le R, \forall t \in [\gamma, T], \\ &\|u^k(\cdot,\gamma)\|_{L_p(\gamma,T;L_p(\Omega))} \le R. \end{aligned} \tag{5.30}$$

For the difference $v^k(t,\gamma) = u^k(t,\gamma) - u(t)$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|v^k(t,\gamma)\|^2 + \beta\|\nabla v^k(t,\gamma)\|^2 + \int_\Omega \big((f^k(t,u^k),u^k) + (f(t,u),u)\big)\,dx \\ &\quad \le \int_\Omega \big((f(t,u),u^k) + (f^k(t,u^k),u)\big)\,dx, \end{aligned} \tag{5.31}$$

for a.a. $t \in (\gamma, T)$. Using conditions (5.9), (5.25) and integrating (5.31) over (γ,γ_0), we obtain

$$\begin{aligned} \|u^k(\gamma_0,\gamma) - u(\gamma_0)\|^2 \le{}& \|u(\gamma) - u(\gamma)\|^2 + K\big[(\gamma_0 - \gamma) \\ &+ \|f(t,u)\|_{L_q(\gamma,\gamma_0;L_q(\Omega))}\|u^k\|_{L_p(\gamma,\gamma_0;L_p(\Omega))} \\ &+ \|f^k(t,u^k)\|_{L_q(\gamma,\gamma_0;L_q(\Omega))}\|u\|_{L_p(\gamma,\gamma_0;L_p(\Omega))}\big]. \end{aligned}$$

It follows from (5.30) and (5.25) that $\|u^k\|_{L_p(\gamma,\gamma_0;L_p(\Omega))}$ and $\|f^k(t,u^k)\|_{L_q(\gamma,\gamma_0;L_q(\Omega))}$ are bounded by a constant that does not depend on γ. On the other hand, $u \in L_p(\gamma,\gamma_0;L_p(\Omega))$ and $f(t,u) \in L_q(\gamma,\gamma_0;L_q(\Omega))$ (by (5.8)), so that

$$\|f(t,u)\|_{L_q(\gamma,\gamma_0;L_q(\Omega))} \le \varepsilon, \quad \|u\|_{L_p(\gamma,\gamma_0;L_p(\Omega))} \le \varepsilon,$$

as soon as $|\gamma - \gamma_0| < \delta(\varepsilon)$. Therefore, $\|u^k(\gamma_0,\gamma) - u(\gamma_0)\| \to 0$, as $\gamma \nearrow \gamma_0$. □

Now, we put

$$\gamma(\lambda) = \begin{cases} \tau - (T-\tau)\lambda, & if\ \lambda \in [-1,0], \\ \tau + (T-\tau)\lambda, & if\ \lambda \in [0,1], \end{cases}$$

and define the function

$$\varphi^k(\lambda)(t) = \begin{cases} u_1^k(t,\gamma(\lambda)), & if\ \lambda \in [-1,0], \\ u_2^k(t,\gamma(\lambda)), & if\ \lambda \in [0,1]. \end{cases}$$

We have $\varphi^k(-1)(t) = u_1^k(t,T) = u_1(t)$, $\varphi^k(1)(t) = u_2^k(t,T) = u_2(t)$. The map $\lambda \mapsto \varphi^k(\lambda)(t) \in H$ is continuous for any fixed $k \ge 1$, $t \in [\tau, T]$ (note that $u_1^k(t,\tau) = u_2^k(t,\tau)$) and $\varphi^k(-1)(t^*) \in U_1$, $\varphi^k(1)(t^*) \in U_2$, so that there exists $\lambda_k \in [-1,1]$ such that $\varphi^k(\lambda_k)(t^*) \notin U_1 \cup U_2$.

Denote $u^k(t) = \varphi^k(\lambda_k)(t)$. Note that for each $k \ge 1$, either $u^k(t) = u_1^k(t,\gamma(\lambda_k))$ or $u^k(t) = u_2^k(t,\gamma(\lambda_k))$. For some subsequence, it is equal to one of them, say $u_1^k(t,\gamma(\lambda_k))$. Now, we shall consider the function $u_1^k(t,\gamma(\lambda_k))$, $t \in [\tau, T]$. We have

$$u^k(t) = \begin{cases} u_1(t), & if\ t \in [\tau, \gamma(\lambda_k)], \\ u_1^k(t, \gamma(\lambda_k)), & if\ t \in [\gamma(\lambda_k), T], \end{cases}$$

where $\gamma(\lambda_k) \to \gamma_0 \in [\tau, T]$. We define the function

$$\widetilde{f}^k(t, v) = \begin{cases} f(t, v), & if\ t \in [\tau, \gamma(\lambda_k)], \\ f^k(t, v), & if\ t \in (\gamma(\lambda_k), T], \end{cases}$$

By continuity, $u_1(\gamma(\lambda_k)) \to u_1(\gamma_0)$, $k \to \infty$. Moreover, from (5.29) and (5.25), the sequence $\{u^k(\cdot)\}$ is bounded in $W_{\tau,T}$, and $\{\widetilde{f}^k(t, u^k)\}$ is bounded in $L_q(\tau, T; L_q(\Omega))$. It follows also that $\{\frac{du^k}{dt}(\cdot)\}$ is bounded in $L_q(\tau, T; H^{-r}(\Omega))$, where $r_i = \max\{1; (\frac{1}{2} - \frac{1}{p_i})N\}$. Arguing as in the proof of Lemma 5.2, we have that for some function $u = u(t, x)$:

$$u^k \to u \text{ in } L_2(\tau, T; H),\ u^k(t) \to u(t) \text{ in } H \text{ for a.a. } t \in (\tau, T), \tag{5.32}$$

$$u^k(t, x) \to u(t, x) \text{ for a.a. } (t, x) \in (\tau, T) \times \Omega, \tag{5.33}$$

$$u^k(t) \to u(t) \text{ weakly in } H, \tag{5.34}$$

$$\frac{du^k}{dt} \to \frac{du}{dt} \text{ weakly in } L_q(\tau, T; H^{-r}(\Omega)). \tag{5.35}$$

Moreover, $\widetilde{f}^k(t, u^k(t, x)) \to f(t, u(t, x))$ for a.a. $(t, x) \in (\tau, T) \times \Omega$, and then the boundedness of $\widetilde{f}^k(t, u^k)$ in $L_q(\tau, T; L_q(\Omega))$ implies that $\widetilde{f}^k(t, u^k(t))$ converges to $f(t, u(t))$ weakly in $L_q(\tau, T; L_q(\Omega))$ (see Lemma 5.1). It follows as in the proof of Lemma 5.2 that $u(\cdot)$ is a weak solution of (5.7) and $u(\tau) = u_\tau$.

Lemma 5.5. *We have*

$$u^k(t^*) \to u(t^*) in\ H.$$

Proof. From (5.11) and (5.29), we have

$$\begin{aligned} \|u^k(t)\|^2 &\le \|u^k(s)\|^2 + D \int_s^t (\|h(v)\|^2 + 1) dv, \\ \|u(t)\|^2 &\le \|u(s)\|^2 + D \int_s^t (\|h(v)\|^2 + 1) dv, \end{aligned} \tag{5.36}$$

for all $t \ge s$, $t, s \in [\tau, T]$, where the constant $D > 0$ does not depend on k.

From (5.36), the functions $J_k(t) = \|u^k(t)\|^2 - D \int_\tau^t (\|h(s)\|^2 + 1) ds$, $J(t) = \|u(t)\|^2 - D \int_\tau^t (\|h(s)\|^2 + 1) ds$ are continuous and nonincreasing on $[\tau, T]$.

Moreover, since $u^k(t) \to u(t)$ in H for a.a. $t \in (\tau, T)$, we have $J_k(t) \to J(t)$ for a.a. $t \in (\tau, T)$. We state that $\limsup J_k(t^*) \leq J(t^*)$. Indeed, let $\tau < t_m < t^*$ be such that $J_k(t_m) \to J(t_m)$. Since J_k are nonincreasing, we obtain

$$J_k\left(t^*\right) - J\left(t^*\right) \leq |J_k(t_m) - J(t_m)| + \left|J(t_m) - J\left(t^*\right)\right|.$$

For any $\varepsilon > 0$, there exist $t_m(\varepsilon)$ and $k_0(t_m)$ such that $J_k(t^*) - J(t^*) \leq \varepsilon$, for all $k \geq k_0$, and the result follows. Hence,

$$\limsup J_k(t^*) = \limsup \|u^k(t^*)\|^2 - D\int_{\tau}^{t^*} (\|h(s)\|^2 + 1)ds \leq$$

$$\leq \|u(t^*)\|^2 - D\int_{\tau}^{t^*} (\|h(s)\|^2 + 1)ds.$$

Therefore, $\limsup \|u^k(t^*)\| \leq \|u(t^*)\|$. Since $u^k(t) \to u(t)$ weakly in H, we have $\liminf \|u^k(t^*)\| \geq \|u(t^*)\|$. Thus, $u^k(t^*) \to u(t^*)$ in H. □

From this, we immediately obtain that $u(t^*) \notin U_1 \cup U_2$, which is a contradiction. □

In applications, it is often necessary to consider the case of nonnegative variables u^i. Denote $\mathbf{R}_+^d = \left\{u \in \mathbf{R}^d : u^i \geq 0\right\}$ and define the space

$$H^+ = \left\{u \in H : u^i(x) \geq 0, \forall i, \text{ for a.e. } x \in \Omega\right\}$$

and the sets

$$\begin{aligned} D_{\tau,T}^+(u_\tau) &= \left\{u(\cdot) : u(\cdot) \in D_{\tau,T}(u_\tau) \text{ and } u(t) \in H^+, \text{ for all } t\right\}, \\ K_t^+(u_\tau) &= \{u(t) : u(\cdot) \in D_{\tau,T}^+(u_\tau)\}. \end{aligned}$$

We shall prove that under additional conditions for any $u_\tau \in H^+$, there exists at least one weak solution $u(\cdot) \in D_{\tau,T}(u_\tau)$ such that $u(t) \in H^+$, for all t. Hence, the sets $D_{\tau,T}^+(u_\tau)$ and $K_t^+(u_\tau)$ are nonempty, and we know by Lemma 5.2 that they are compact. Moreover, we prove that $K_t^+(u_\tau)$ is connected in H.

Theorem 5.4. *Let (5.8)–(5.9) hold for $u \in \mathbf{R}_+^d$. Assume also the following conditions:*

$$\text{The matrix } a \text{ is diagonal;} \tag{5.37}$$

$$\begin{aligned} &h^i(t,x) - f^i\left(t, u^1, \ldots, u^{i-1}, 0, u^{i+1}, \ldots, u^d\right) \geq 0, \\ &\text{for all } i, \text{ a.e. } x \in \Omega, t \in (\tau, T), \text{ and } u^j \geq 0 \text{ if } j \neq i. \end{aligned} \tag{5.38}$$

Then for all $t \in [\tau, T]$ and $u_\tau \in H^+$, the set $K_t^+(u_\tau)$ is nonempty and connected.

Proof. We shall prove first that the set $K_t^+(u_\tau)$ is nonempty.

For every $n \geq 1$, we put $f_n^i(t,u) = \psi_n(|u|) f^i(t,u) + (1-\psi_n(|u|)) g^i(t,u)$, where $g^i(t,u) = \left|u^i\right|^{p_i-2} u^i + f^i(t,0,\ldots,0)$, and ψ_n was defined in (5.22). Then, $f_n \in \mathbf{C}([\boldsymbol{\tau},\mathbf{T}] \times \mathbf{R}^d; \mathbf{R}^d)$ and for any $A > 0$,

$$\sup_{t\in[\tau,T]} \sup_{|u|\leq A} |f_n(t,u) - f(t,u)| \to 0, \; as\; n \to \infty.$$

We can easily check as in Lemma 5.3 that f_n satisfy conditions (5.8)–(5.9) for $u \in \mathbf{R}_+^d$, where the constants do not depend on n. Also, it follows from the conditions of the theorem that

$$h^i(t,x) - f_n^i(t,u) = \psi_n(|u|)\left(h^i(t,x) - f^i(t,u)\right) + (1-\psi_n(|u|))\left(h^i(t,x) - f^i(t,0,\ldots,0)\right) \geq 0,$$

for all t, i, a.e. $x \in \Omega$ and u such that $u^i = 0$ and $u^j \geq 0$ if $j \neq i$. Moreover, if $|u| > n+1$, then for any $w \in \mathbf{R}^d$,

$$(f_{nu}(t,u)w,w) = (g_u(t,u)w,w) = \sum_{i=1}^{d} (p_i - 1)\left|u^i\right|^{p_i-2} w_i^2 \geq 0. \tag{5.39}$$

For every $n \geq 1$, consider the sequence $f_n^\varepsilon(t,u)$ defined by

$$f_n^\epsilon(t,u) = \int_{\mathbf{R}^d} \rho_\epsilon(s) f_n(t,u-s)ds.$$

Since any f_n are uniformly continuous on $[\tau,T] \times [-k-1,k+1]$, for any $k \geq 1$, there exist $\epsilon_{k,n} \in (0,1)$ such that for all u satisfying $|u| \leq k$, and for all s for which $|u-s| < \epsilon_{k,n}$, we have

$$\sup_{t\in[\tau,T]} |f_n(t,u) - f_n(t,s)| \leq \frac{1}{k}.$$

We put $f_n^k(t,u) = f_n^{\epsilon_{k,n}}(t,u)$. Then $f_n^k(t,\cdot) \in \mathbf{C}^\infty(\mathbf{R}^d;\mathbf{R}^d)$, for all $t \in [\tau,T]$, $k,n \geq 1$. Since for any compact subset $A \subset \mathbf{R}^d$ and any n we have $f_n^k \to f_n$ uniformly on $[\tau,T] \times A$, we obtain the existence of a sequence $\delta_{nk} \in (0,1)$ such that $\delta_{nk} \to 0$, as $k \to \infty$, and $\left|f_n^{ki}(t,u) - f_n^i(t,u)\right| \leq \delta_{nk}$, for any i, n and any u satisfying $|u| \leq n+2$. Then the function $F_n^k = \left(F_n^{k1},\ldots,F_n^{kd}\right)$ given by

$$F_n^{ki}(t,u) = f_n^{ki}(t,u) - \delta_{nk}$$

satisfies $h^i(t,x) - F_n^{ki}(t,u) \geq 0$, for u such that $u_i = 0$, $u_j \geq 0$, $j \neq i$, and $|u| \leq n+2$.

Define a smooth function $\phi_n : \mathbf{R}_+ \to [0,1]$ satisfying

$$\phi_n(s) = \begin{cases} 1, \text{ if } 0 \le s \le n+1+\gamma, \\ 0 \le \phi_n(s) \le 1, \text{ if } n+1+\gamma \le s \le n+2, \\ 0, \text{ if } s \ge n+2, \end{cases}$$

where $0 < \gamma < 1$ is fixed. Let $l_n^k(t,u)$ be given by

$$l_n^k(t,u) = \phi_n(|u|) F_n^k(t,u) + (1 - \phi_n(|u|)) f_n(t,u).$$

Then if u is such that $u_i = 0, u_j \ge 0, j \ne i$, then we have

$$\phi_n(|u|) \left(h^i(t,x) - F_n^{ki}(t,u)\right) \ge 0, \ (1 - \phi_n(|u|)) \left(h^i(t,x) - f_n^i(t,u)\right) \ge 0,$$

as $\phi_n(|u|) = 0$, for $|u| \ge n+2$. Hence, $h^i(t,x) - l_n^{ki}(t,u) \ge 0$.

On the other hand, arguing as in the proof of Lemma 5.3, we can prove the existence of C_1, C_2, and α (not depending neither on n or k) such that F_n^k satisfy (5.8)–(5.9) for $u \in \mathbf{R}_+^d$, and then it follows easily that $l_n^k(t,u)$ also satisfies (5.8)–(5.9). It is also clear that $l_n^k(t,u)$ is continuously differentiable with respect to u for any t and u. We obtain the existence of $D_4(k,n)$ such that

$$(l_{nu}^k(t,u)w, w) \ge -D_4(k,n)|w|^2, \text{ for all } w \in \mathbf{R}^d.$$

Indeed, if $|u| \le n+1+\gamma$, then $l_n^k(t,u) = F_n^k(t,u)$, so that

$$(l_{nu}^k(t,u)w, w) = (F_{nu}^k(t,u)w, w) = (f_{nu}^k(t,u)w, w) \ge -D_3(k,n)|w|^2.$$

The last inequality can be proved in the same way as in the proof of Lemma 5.3.

If $|u| \ge n+2$, then $l_n^k(t,u) = f_n(t,u)$, so $(l_{nu}^k(t,u)w, w) = (f_{nu}(t,u)w, w) \ge 0$. Finally, if $n+1+\gamma < |u| < n+2$, we have

$$\begin{aligned}(l_{nu}^k(t,u)w, w) &= \phi_n(|u|)(F_{nu}^k(t,u)w, w) + (1 - \phi_n(|u|))(f_{nu}(t,u)w, w) \\ &\quad + \sum_{i,j=1}^d \frac{\partial}{\partial u_j}\phi_n(|u|) F_n^{ki}(t,u) w_i w_j - \sum_{i,j=1}^d \frac{\partial}{\partial u_j}\phi_n(|u|) f_n^i(t,u) w_i w_j \\ &\ge -D_4(k,n)|w|^2,\end{aligned}$$

where we have used similar arguments as in the proof of Lemma 5.3, (5.39) and also that

$$\left|\frac{\partial}{\partial u_j}\phi_n(|u|)\right| \le R_1(n), \left|f_n^i(t,u)\right| \le R_2(n), \left|F_n^{ki}(t,u)\right| \le R_3(k,n),$$

for any u satisfying $n+1+\gamma < |u| < n+2$, $t \in [\tau, T]$ and any i, j.

Let us consider the approximate problem

$$\begin{cases} \frac{\partial u}{\partial t} - a\Delta u + l_n^k(t,u) = h(t,x), & (t,x) \in (\tau, T) \times \Omega, \\ u|_{x\in\partial\Omega} = 0, \ u|_{t=\tau} = u_\tau. \end{cases} \tag{5.40}$$

System (5.40) has a unique solution that will be denoted by $u_n^k(t)$. It is well known (see Lemma 5.6 below) that $u_n^k(t) \in H^+$, for all $t \geq 0$.

Repeating the same arguments of the proof of Theorem 5.3, we obtain that (up to a subsequence) u_n^k converges to a weak solution u_n of (5.7) (where we replace f by f_n) in the sense of (5.32)–(5.35) and also that $u_n^k(t) \to u_n(t)$ in H for any t. Therefore, $u_n(t) \in H^+$, for all $t \geq 0$.

After that, repeating the same procedure as $n \to \infty$, we obtain that u_n converges to a weak solution u of (5.7) and $u(t) \in H^+$, for all $t \geq 0$. Then we have that the limit function u belongs to $\mathscr{D}_{\tau,T}^+(u_\tau)$. Hence, $K_t^+(u_\tau)$ is nonempty.

The connectedness of the set $K_t^+(u_\tau)$ is proved in a similar way as in the proof of Theorem 5.3. □

Lemma 5.6. *Let (5.8)–(5.9) hold for $u \in \mathbf{R}_+^d$. Assume also that (5.37)–(5.38) hold and that $f(t,u)$ is continuously differentiable with respect to u for any $t \in [\tau, T]$, $u \in \mathbf{R}^d$, and*

$$(f_u(t,u)w, w) \geq -C_3(t)|w|^2, \textit{for all } w, u \in \mathbf{R}^d, \tag{5.41}$$

where $C_3(\cdot) \in L_1(\tau, T)$, $C_3(t) \geq 0$. Then for any $u_\tau \in H^+$, there exists a unique weak solution $u(\cdot)$ of (5.7), which satisfies $u(t) \geq 0$ for all $t \in [\tau, T]$.

Proof. Condition (5.41) implies in a standard way that at most one solution can exist (see the proof of Theorem 5.3). Let us prove that this solution should be non-negative.

Let $u^+ = \max\{u, 0\}$. Define the functions $g^i(t,x,u) = h^i(t,x) - f^i(t,u) - C_3(t)u^i$. It follows from the condition of the theorem that $g^i(t,x,u) \geq 0$ if $u^i = 0$ and $u^j \geq 0$ for $j \neq i$. Let us consider the scalar product

$$\left(g(t,x,u), (-u)^+\right) = \sum_{i=1}^d g^i(t,x,u)\left(-u^i\right)^+.$$

Let $J \subset I = \{1, \ldots, d\}$ be such that $u^i < 0$, for $i \in J$, and $u^i \geq 0$, for $i \in I \backslash J$. We denote the number of elements of J by $\overline{d} \leq d$. Let $\overline{g} : [\tau, T] \times \Omega \times \mathbf{R}^d \to \mathbf{R}^{\overline{d}}$ be defined by $\overline{g} = \left(g^i\right)_{i\in J}$. It is clear that

$$\left(g(t,x,u), (-u)^+\right) = \sum_{i\in J} g^i(t,x,u)\left(-u^i\right)^+ = \left(\overline{g}(t,x,u), (-u)^+\right).$$

Also, as $g^i\left(t,x,u^+\right) \geq 0$ if $i \in J$, and $u - u^+ = -(-u)^+$, the Mean Value Theorem and (5.41) imply

$$\begin{aligned}\sum_{i\in J} g^i(t,x,u)\left(-u^i\right)^+ &\geq \sum_{i\in J}\left(g^i(t,x,u) - g^i\left(t,x,u^+\right)\right)\left(-u^i\right)^+\\ &= -\sum_{i\in J}\left(g_u^i(t,x,v(t,x,u)),(-u)^+\right)\left(-u^i\right)^+\\ &= -\sum_{i=1}^{d}\left(g_u^i(t,x,v(t,x,u)),(-u)^+\right)\left(-u^i\right)^+\\ &= -\left(g_u(t,x,v(t,x,u))(-u)^+,(-u)^+\right) \geq 0.\end{aligned}$$

We multiply the equation by $(-u)^+$. As $(-u)^+ \in L_2(\tau,T;V)$ and $\frac{1}{2}\frac{d}{dt}\left\|(-u)^+\right\|^2 = -\left\langle \frac{du}{dt},(-u)^+\right\rangle$ [5, p.203], we have

$$-\frac{1}{2}\frac{d}{dt}\left\|(-u)^+\right\|^2 - \beta\left\|(-u)^+\right\|_V^2 \geq \int_\Omega \left(g(t,x,u),(-u)^+\right)dx + C_3(t)\left(u,(-u)^+\right),$$

so $\frac{d}{dt}\left\|(-u)^+\right\|^2 \leq C_3(t)\left\|(-u)^+\right\|^2$. By Gronwall's lemma, we get $\left\|(-u)^+(t)\right\| = 0$, which means that $u(x,t) \geq 0$, for a.a. $x \in \Omega$ and all $t \in [\tau,T]$.

Finally, we prove the existence of such solution. For every $n \geq 1$, we put $f_n^i(t,u) = \psi_n(|u|)f^i(t,u) + (1-\psi_n(|u|))g^i(t,u)$, where $g^i(t,u) = \left|u^i\right|^{p_i-2}u^i + f^i(t,0,\ldots,0)$, as in the proof of Theorem 5.4. We have seen there that f_n satisfies conditions (5.8)–(5.9) for $u \in \mathbf{R}_+^d$, where the constants do not depend on n, and also that $h^i(t,x) - f_n^i(t,u) \geq 0$, for all t,i, a.e. $x \in \Omega$ and u such that $u^i = 0$ and $u^j \geq 0$ if $j \neq i$. Moreover, arguing as in the proof of Theorem 5.4, one can check that for any $w \in \mathbf{R}^d$, $(f_{nu}(t,u)w,w) \geq -D(n)|w|^2$. It is easy to see that (5.8)–(5.9) are also satisfied for any $u \in \mathbf{R}^d\backslash\mathbf{R}_+^d$, although in this case the constants can depend on n. Hence, the existence and uniqueness of weak solutions in this case is a well-known result in the literature (see, e.g., [4]). As we have seen before, this solution satisfies $u_n(t) \in H^+$, for all $t \geq 0$. By this property, we can obtain estimates of the solution which are independent of n (as (5.8)–(5.9) hold for $u \in \mathbf{R}_+^d$ with constant independent of n), and then repeating the same arguments of the proof of Theorem 5.3, we obtain that (up to a subsequence) u_n converges to the unique weak solution u of (5.7) in the sense of (5.32)–(5.35) and also that $u_n(t) \to u(t)$ in H for any t. Hence, $u(t) \in H^+$ for all $t \geq 0$. □

Remark 5.1. The results of this section remain valid if we change the Dirichlet boundary conditions by Neumann ones ($\frac{\partial u}{\partial \nu}|_{x\in\partial\Omega} = 0$). In this case, $V = \left(H^1(\Omega)\right)^d$.

We could consider also the case where the nonlinear function f can depend explicitly on $x \in \Omega$. In order to obtain the Kneser's property, we need to assume additionally that $f = \left(f^1(x,t,u),\ldots,f^d(x,t,u)\right)$ is jointly continuous on $[\tau,T]\times \mathbf{R}^d$ uniformly with respect to $x \in \Omega$ and also that it is measurable on x for all (t,u). Moreover, the constants in conditions (5.8)–(5.9) do not depend on $x \in \Omega$.

Then arguing in a similar way, we can prove that the statement of Theorems 5.3 and 5.4 remain valid in this more general case.

We can give an example for which we know that at least two solutions corresponding to a given initial data exist. Let $\lambda_1 > 0$ be the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ and ψ_1 be the corresponding eigenfunction. Without loss of generality, we can assume that $\psi_1(x) > 0$, for any $x \in \Omega$. It is known that $\psi_1 \in \mathbf{C}(\overline{\Omega})$, so that $\max\limits_{x\in\overline{\Omega}} |\psi_1(x)| \le K$. We put

$$f(x,u) = \begin{cases} -\lambda_1 u - \sqrt{\psi_1(x)}\sqrt{u}, \text{ if } u \in [0,1], \\ -\lambda_1 u - \sqrt{\psi_1(x)}u + u^2(u-1), \text{ if } u \notin [0,1], \end{cases} \tag{5.42}$$

and $h \equiv 0$. It is easy to check that $f(x,u)$ satisfies conditions (5.8)–(5.9) with $p = 4$. Suppose that $a = 1,\ \tau = 0, u_0 = 0$. Then $u(t,x) \equiv 0$ is a trivial solution of the Cauchy problem (5.7). For fixed $r \ge 0$, we define

$$u_r(t,x) = \begin{cases} 0,\ 0 \le t \le r, \\ \frac{1}{4}(t-r)^2\psi_1(x),\ r \le t \le r + \frac{2}{\sqrt{K}}, \\ v_r(t,x), \text{if } t_r \le t \le T, \end{cases}$$

where $t_r = r + \frac{2}{\sqrt{K}}$ and $v_r(t,x)$ is a solution on $[t_r,T]$ with $v_r(t_r,x) = \frac{\psi_1(x)}{K}$. It is easy to see that till $\frac{1}{4}(t-r)^2\psi_1(x) \le 1$, the function $u_\tau(t,x)$ is a solution of (5.7), and it is clear that $|u_r(t,x)| \le 1$, for all $t \in [r,t_\tau]$. Hence, $u_r(t,x)$ is another solution of problem (5.7), as the concatenation of solutions is again a solution (see Lemma 5.8 below).

5.2.1 Application to the Complex Ginzburg–Landau Equation

Let us consider (5.5).

For $v = \left(u^1,u^2\right)$, $u = u^1 + iu^2$, (5.5) can be written as the system

$$\frac{\partial v}{\partial t} = \begin{pmatrix} 1 & -\eta \\ \eta & 1 \end{pmatrix} \Delta v + \begin{pmatrix} R(t)\,u^1 - \left(\left|u^1\right|^2 + \left|u^2\right|^2\right)\left(u^1 - \beta(t)\,u^2\right) \\ R(t)\,u_2 - \left(\left|u^1\right|^2 + \left|u^2\right|^2\right)\left(\beta(t)\,u^1 + u^2\right) \end{pmatrix} + \begin{pmatrix} g^1(t,x) \\ g^2(t,x) \end{pmatrix}$$

and conditions (5.8)–(5.9) hold with $p = (4,4)$.

Indeed, as

$$f(t,v) = \left(-R(t)\,u^1 + |v|^2\left(u^1 - \beta(t)\,u_2\right), -R(t)\,u_2 + |v|^2\left(\beta(t)\,u^1 + u_2\right)\right),$$

we have by using the Young's inequality that

$$\left|f^1(t,v)\right|^{\frac{4}{3}} + \left|f^2(t,v)\right|^{\frac{4}{3}} \leq K_1\left(|R(t)|^{\frac{4}{3}}\left(\left|u^1\right|^{\frac{4}{3}} + \left|u^2\right|^{\frac{4}{3}}\right)\right.$$
$$\left. + |v|^{\frac{8}{3}}\left(1 + |\beta(t)|^{\frac{4}{3}}\right)\left(\left|u^1\right|^{\frac{4}{3}} + \left|u^2\right|^{\frac{4}{3}}\right)\right) \leq K_2\left(\left|u^1\right|^4 + \left|u^2\right|^4\right) + K_3,$$

where we have used that $R(t)$, $\beta(t)$ are uniformly bounded in $[\tau, T]$. Also,

$$(f(t,v), v) = -R(t)|v|^2 + |v|^4 \geq \frac{|v|^4}{2} - K_4 \geq \frac{\left|u^1\right|^4 + \left|u^2\right|^4}{2} - K_4.$$

Hence, from Lemma 5.2 and Theorem 5.3, it follows the following:

Theorem 5.5. *For (5.5), the attainability set $K_t(u_\tau)$ is compact and connected for any $t \in [\tau, T]$.*

5.2.2 Application to the Lotka–Volterra System with Diffusion

Let us consider now system (5.6). In this case, the function f is given by

$$f(t,u) = \begin{pmatrix} -u^1\left(a_1(t) - u^1 - a_{12}(t)u^2 - a_{13}(t)u^3\right) \\ -u^2\left(a_2(t) - u^2 - a_{21}(t)u^1 - a_{23}(t)u^3\right) \\ -u^3\left(a_3(t) - u^3 - a_{31}(t)u^1 - a_{32}(t)u^2\right) \end{pmatrix}.$$

Then conditions (5.8)–(5.9) hold for $u \in \mathbf{R}^3_+$ with $p = (3,3,3)$. Indeed, as $u^i \geq 0$, using the Young's inequality, we have

$$\begin{aligned}(f(t,u), u) &\geq \left(u^1\right)^3 + \left(u^2\right)^3 + \left(u^3\right)^3 - a_1(t)\left(u^1\right)^2 - a_2(t)\left(u^2\right)^2 - a_3(t)\left(u^3\right)^2 \\ &\geq \tfrac{1}{2}\left(\left(u^1\right)^3 + \left(u^2\right)^3 + \left(u^3\right)^3\right) - K_1,\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{3}|f_i(t,u)|^{\frac{3}{2}} \leq K_2\Big(&\left(u^1\right)^3 + \left(u^2\right)^3 + \left(u^3\right)^3 + \left(u^1\right)^{\frac{3}{2}} + \left(u^2\right)^{\frac{3}{2}} + \left(u^3\right)^{\frac{3}{2}} \\ &+ \left(u^1u^2\right)^{\frac{3}{2}} + \left(u^2u^3\right)^{\frac{3}{2}} + \left(u^1u^3\right)^{\frac{3}{2}}\Big) \\ \leq K_3&\left(\left(u^1\right)^3 + \left(u^2\right)^3 + \left(u^3\right)^3\right) + K_4.\end{aligned}$$

It is clear that (5.37)–(5.38) are satisfied. Hence, from Lemma 5.2, Theorem 5.4, and Remark 5.1, it follows the following:

Theorem 5.6. *For equation (5.6), the attainability set $K_t^+(u_\tau)$ is compact and connected for any $t \in [\tau, T]$.*

5.3 Connectedness of Attractors for Reaction-Diffusion Systems

Let us consider again the system of reaction-diffusion (5.7).

We have proved in Theorems 5.3, 5.4 that the solutions of this system satisfies the Kneser's property. Now, we shall use this property in order to check that its global attractor is connected.

We shall define a multivalued process (MP) as given in Chap. 4. For this aim, we need some additional conditions. As before, we assume that (5.8)–(5.9) hold with constants not depending on $t \in \mathbf{R}$. Suppose now that $h \in L_2^{loc}(\mathbf{R}; H)$ and

$$\|h\|_b^2 = \sup_{t \in \mathbf{R}} \int_t^{t+1} \|h(s)\|^2 \, ds < \infty. \tag{5.43}$$

We define the hull $\mathscr{H}(h) = Cl_Y \{h(\cdot + s) : s \in \mathbf{R}\}$, where Cl_Y denotes the closure in the space Y and $Y = L_{2,w}^{loc}(\mathbf{R}; H)$, that is, Y is the space $L_2^{loc}(\mathbf{R}; H)$ endowed with the weak topology. Condition (5.43) implies that $\mathscr{H}(h)$ is compact in Y [4, p.105].

We shall assume also that the function $f : \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}^d$ satisfies

$$|f(t,u) - f(s,v)| \le \omega(|t-s| + |u-v|, K), \text{ for all } t, s \in \mathbf{R}, |u|, |v| \le K, K>0, \tag{5.44}$$

where $\omega(l, K) \to 0$, as $l \to 0^+$. We put

$$U = \left\{ \psi \in C\left(\mathbf{R}^d, \mathbf{R}^d\right) : \sum_{i=1}^d \left|\psi^i(u)\right|^{\frac{p_i}{p_i-1}} \le C\left(1 + \sum_{i=1}^d \left|u^i\right|^{p_i}\right)\right\},$$

$$\|\psi\|_U = \sum_{i=1}^d \sum_{j=1}^\infty \alpha_j \sup_{|u| \le K_j} \frac{\left|\psi^i(u)\right|}{\left(1 + \sum_{k=1}^d \left|u^k\right|^{p_k}\right)^{\frac{p_i-1}{p_i}}},$$

where K_j, α_j are positive sequences such that $K_j \to +\infty$ and $\sum_{j=1}^\infty \alpha_j < \infty$. U is a Banach space. By an abuse of notation from any fixed s, we denote by $f(s)$ the element ξ_s of U given by $\xi_s(u) = f(s,u)$. Since $f(t,u)$ is jointly continuous on $\mathbf{R} \times \mathbf{R}^d$, (5.8) gives $f(\cdot) \in C(\mathbf{R}, U)$. It follows from (5.44) that the hull $\mathscr{H}(f) = Cl_Z \{f(\cdot + s) : s \in \mathbf{R}\}$ is compact, where $Z = C(\mathbf{R}, U)$ [4, p.101].

Let $W = Y \times Z$. We put $\Sigma = Cl_W \{\sigma_0(\cdot + s) : s \in \mathbf{R}\}$, where $\sigma_0(s) = (h(s), f(s))$, which is a compact set in $Y \times Z$. It is clear that $T(s)\Sigma \subset \Sigma$, where $T(s)\sigma = \sigma(\cdot + s) = (h_\sigma(\cdot + s), f_\sigma(\cdot + s))$, $s \in \mathbf{R}$, and that this map is continuous, so that $(Z1)-(Z2)$ are satisfied. Moreover, it is evident that if (5.8)–(5.9) hold with constants not depending on t, and (5.43)–(5.44) are satisfied, then for any $\sigma = (h_\sigma, f_\sigma) \in \Sigma$, we have that f_σ satisfies also (5.8)–(5.9) and (5.44) with the same constants C_1, C_2, α and the same function ω. Also, $\|h_\sigma\|_b \leq \|h\|_b$, so that (5.43) holds. Moreover, we have:

Lemma 5.7. *If (h, f) satisfies (5.38) in any interval $(\tau, T) \subset \mathbf{R}$, then every $(h_\sigma, f_\sigma) \in \Sigma$ also satisfies (5.38) in any (τ, T).*

Proof. It is clear that for a.a. $(t, x) \in (\tau, T) \times \Omega$ and any s, the functions $(h_\sigma, f_\sigma) = (h(\cdot + s), f(\cdot + s))$ satisfy (5.38). Let $(h_\sigma, f_\sigma) = \lim_{n\to\infty} (h(\cdot + s_n), f(\cdot + s_n))$ in W, with $s_n \to \pm\infty$. Let $B_\varepsilon(x_0) = \{x \in \Omega : |x - x_0| < \varepsilon\}$. For a.a. $(t_0, x_0) \in (\tau, T) \times \Omega$, taking a sufficiently small $\varepsilon > 0$, we have

$$\frac{1}{2\varepsilon}\frac{1}{\mu(B_\varepsilon(x_0))}\int_{t_0-\varepsilon}^{t_0+\varepsilon}\int_{B_\varepsilon(x_0)} h_\sigma^i(t,x)\,dxdt \to h_\sigma^i(t_0,x_0), \text{ for any } i. \tag{5.45}$$

Denote $(h_n(\cdot), f_n(\cdot)) = (h(\cdot + s_n), f(\cdot + s_n))$. Let (t_0, x_0) be a point where (5.45) holds and (5.38) is satisfied for all (h_n, f_n). Let $\eta(t,x) = I_{B_\varepsilon(x_0)}(x)$ be the indicator function on the ball $B_\varepsilon(x_0)$, which belongs to $L_2(\tau, T; L_2(\Omega))$. Then

$$\int_{t_0-\varepsilon}^{t_0+\varepsilon}\int_\Omega h_n^i(t,x)\,\eta(t,x)\,dxdt \to \int_{t_0-\varepsilon}^{t_0+\varepsilon}\int_\Omega h_\sigma^i(t,x)\,\eta(t,x)\,dxdt,$$

so

$$\int_{t_0-\varepsilon}^{t_0+\varepsilon}\int_\Omega h_\sigma^i(t,x)\,\eta(t,x)\,dxdt \geq \mu(B_\varepsilon(x_0))\int_{t_0-\varepsilon}^{t_0+\varepsilon} f_\sigma^i(t,u)\,dt,$$

if $u^i = 0$, and $u^j \geq 0$ if $j \neq i$. Hence,

$$\frac{1}{2\varepsilon}\frac{1}{\mu(B_\varepsilon(x_0))}\int_{t_0-\varepsilon}^{t_0+\varepsilon}\int_{B_\varepsilon(x_0)} h_\sigma^i(t,x)\,dxdt \geq \frac{1}{2\varepsilon}\int_{t_0-\varepsilon}^{t_0+\varepsilon} f_\sigma^i(t,u)\,dt.$$

Passing to the limit as $\varepsilon \to 0$, we obtain that $h_\sigma^i(t_0, x_0) \geq f_\sigma^i(t_0, u)$. □

With respect to the general setting of Chap. 4, we put $X = H = (L_2(\Omega))^d$ endowed with the strong topology.

We have seen before that for every $T > \tau$ and $u_\tau \in H$, the set $D_{\tau,T}(u_\tau)$ of weak solutions of (5.7) is nonempty.

Lemma 5.8. *If $u(\cdot) \in D_{\tau,T}(u_\tau)$, then $v(\cdot) = u(s + \cdot) \in D_{\tau-s,T-s}(u_\tau)$, for any s, if we change $f(t), h(t)$ by $f(t+s), h(t+s)$, respectively. If $u(\cdot) \in D_{\tau,t}(u_\tau)$ and $v(\cdot) \in D_{t,T}(u(t))$, then*

$$z(s) = \begin{cases} u(s), \text{ if } s \in [\tau, t], \\ v(s), \text{ if } s \in [t, T], \end{cases}$$

belongs to $D_{\tau,T}(u_\tau)$.

Proof. If $u(\cdot) \in \mathscr{D}_{\tau,T}(u_\tau)$, then

$$v(\cdot) = u(s+\cdot) \in W_{\tau-s,T-s},$$
$$\frac{dv}{dt} \in L_2\left(\tau-s, T-s; V'\right) + L_q\left(\tau-s, T-s; L_q(\Omega)\right), v(\tau-s) = u_\tau,$$

and

$$\int_{\tau-s}^{T-s}\left\langle \frac{dv(t)}{dt}, \xi(t)\right\rangle dt + \int_{\tau-s}^{T-s}\int_\Omega (a\nabla v(t), \nabla\xi(t))\, dxdt$$
$$+\int_{\tau-s}^{T-s}\int_\Omega (f(t+s, v(t)), \xi(t))\, dxdt$$
$$= \int_\tau^T \left\langle \frac{du(r)}{dt}, \xi(r-s)\right\rangle dr + \int_\tau^T\int_\Omega (a\nabla u(r), \nabla\xi(r-s)\, dxdr$$
$$+\int_\tau^T\int_\Omega (f(r,u(r)), \xi(r-s))\, dxdr$$
$$= \int_\tau^T\int_\Omega (h(r), \xi(r-s))\, dxdt = \int_{\tau-s}^{T-s}\int_\Omega (h(t+s), \xi(t))\, dxdt,$$

for any $\xi \in L_p(\tau-s, T-s; L_p(\Omega)) \cap L_2(\tau-s, T-s; V)$, so that $v \in D_{\tau-s,T-s}(u_\tau)$.

If $u(\cdot) \in D_{\tau,t}(u_\tau)$ and $v(\cdot) \in D_{t,T}(u(t))$, then the function $z(\cdot)$ belongs to $W_{\tau,T}$. Also, it is easy to see that $z(\cdot)$ is an $H^{-r}(\Omega)$-valued absolutely continuous function in the interval $[\tau, T]$. Hence, $z(\cdot)$ is almost everywhere differentiable and

$$\frac{dz}{dt}(t) = \begin{cases} \frac{du}{dt}(t) & \text{for a.a. } t \in (\tau, t), \\ \frac{dv}{dt}(t) & \text{for a.a. } t \in (t, T). \end{cases}$$

Let Dz be derivative of z in the sense of $H^{-r}(\Omega)$-valued distributions $\mathscr{D}'(0, T; H^{-r}(\Omega))$. Since $\frac{dz}{dt} \in L_q(\tau, T; H^{-r}(\Omega))$, we have that $Dz = \frac{dz}{dt}$ [2]. It follows also that $\frac{dz}{dt} \in L_2(\tau, T; V') + L_q\left(\tau, T; L_q(\Omega)\right)$. Finally,

$$\int_\tau^T \left\langle \frac{dz}{dt}, \xi\right\rangle dr + \int_\tau^T\int_\Omega (a\nabla z, \nabla\xi)\, dxdr + \int_\tau^T\int_\Omega (f(r, z(r)), \xi)\, dxdr$$
$$= \int_\tau^t \left\langle \frac{du}{dt}, \xi\right\rangle dr + \int_\tau^t\int_\Omega (a\nabla u, \nabla\xi)\, dxdr + \int_\tau^t\int_\Omega (f(r, u(r)), \xi)\, dxdr$$
$$+\int_t^T \left\langle \frac{dv}{dt}, \xi\right\rangle dr + \int_t^T\int_\Omega (a\nabla v, \nabla\xi)\, dxdr + \int_t^T\int_\Omega (f(r, v(r)), \xi)\, dxdr$$
$$= \int_\tau^t\int_\Omega (h(r), \xi)\, dxdt + \int_t^T\int_\Omega (h(r), \xi)\, dxdt = \int_\tau^T\int_\Omega (h(r), \xi)\, dxdt.$$

Hence, $z \in D_{\tau,T}(u_\tau)$. □

Hence, any weak solution can be extended to a global one (i.e., defined for $t \geq \tau$). We denote by $D_{\tau,\sigma}(u_\tau)$ the set of all global weak solutions (defined for $t \geq \tau$) of problem (5.7) with data (h_σ, f_σ) instead of (h, f), such that $u(\tau) = u_\tau$. For each $\sigma \in \Sigma$, we define the map:

$$U_\sigma(t, \tau, u_\tau) = \{u(t) : u(\cdot) \in D_{\tau,\sigma}(u_\tau)\}.$$

This map is well defined, as the set $D_{\tau,\sigma}(u_\tau)$ is nonempty for every $\sigma \in \Sigma, \tau \in \mathbf{R}$ and $u_\tau \in H$.

Lemma 5.9. *U_σ is a strict multivalued process and*

$$U_\sigma(t+s, \tau+s, u_\tau) = U_{T(s)\sigma}(t, \tau, u_\tau), \textit{ for all } u_\tau \in H, s \in \mathbf{R}, t \geq \tau,$$

so that L3B holds.

Proof. Let $y \in U_\sigma(t, \tau, u_\tau)$. Then $y = u(t)$, where $u(\cdot) \in D_{\tau,\sigma}(u_\tau)$. Let $\tau \leq s \leq t$. It is easy to see that $u(\cdot) \in D_{s,\sigma}(u(s))$, as well. Then we have $y = u(t) \in U_\sigma(t, s, u(s)) \subset U_\sigma(t, s, U_\sigma(s, , u_\tau))$.

Let now $y \in U_\sigma(t, s, U_\sigma(s, \tau, u_\tau))$. Then there exist $u(\cdot) \in D_{\tau,\sigma}(u_\tau)$ and $v(\cdot) \in D_{s,\sigma}(u(s))$ such that $y = v(t)$. Define the function

$$z(r) = \begin{cases} u(r), \text{ if } r \in [\tau, s], \\ v(r), \text{ if } r \in [s, t]. \end{cases}$$

Lemma 5.8 implies that $z \in D_{\tau,\sigma}(u_0)$, so that $y = z(t) \in U_\sigma(t, \tau, u_\tau)$.

Let $y \in U_\sigma(t+s, \tau+s, u_\tau)$. Then there exists $u(\cdot) \in D_{\tau+s,\sigma}(u_\tau)$ such that $y = u(t+s)$. It follows from Lemma 5.8 that $v(\cdot) = u(s+\cdot) \in D_{\tau,T(s)\sigma}(u_\tau)$, so that $y = v(t) \in U_{T(s)\sigma}(t, \tau, u_\tau)$. Conversely, if $y \in U_{T(s)\sigma}(t, \tau, u_\tau)$, then there is $u(\cdot) \in D_{\tau,T(s)\sigma}(u_\tau)$ such that $y = u(t)$. Again by Lemma 5.8, we have that $v(\cdot) = u(-s+\cdot) \in D_{\tau+s,\sigma}(u_\tau)$, so that $y = v(t+s) \in U_\sigma(t+s, \tau+s, u_\tau)$. □

Theorem 5.7. *Let (5.43)–(5.44) and (5.8)–(5.9) hold (with constants not depending on $t \in \mathbf{R}$). Then the MP U_Σ possesses the uniform global compact invariant attractor Θ_Σ, which is connected in H.*

Proof. It follows from Lemma 5.2 that the family of strict MP U_σ satisfies the following properties:

1. The map $(\sigma, x) \mapsto U_\sigma(t, \tau, x)$ has closed graph for all $t \geq \tau$.
2. The map $(\sigma, x) \mapsto U_\sigma(t, \tau, x)$ is upper semicontinuous for all $t \geq \tau$.
3. U_σ has compact values.

Indeed, in view of Lemma 5.2, U_σ has compact values. Let us prove that the map $(\sigma, x) \mapsto U_\sigma(t, \tau, u_\tau)$ is upper semicontinuous for each fixed $t \geq \tau$. Suppose that it is not true, that is, there exist $u_\tau \in H$, $t \geq \tau$, $\sigma_0 \in \Sigma$, a neighborhood O of $U_\sigma(t, \tau, u_\tau)$, $u_n \to u_\tau$, $\sigma_n \to \sigma_0$, and $\xi_n \in U_{\sigma_n}(t, \tau, u_n)$ such that $\xi_n \notin O$. But

Lemma 5.2 implies that (up to a subsequence) $\xi_n \to \xi \in U_{\sigma_0}(t, \tau, u_\tau)$, which is a contradiction. Finally, if $u_n \to u_\tau$, $\sigma_n \to \sigma_0$, and $\xi_n \in U_{\sigma_n}(t, \tau, u_n)$, $\xi_n \to \xi$, then again Lemma 5.2 implies that $\xi \in U_{\sigma_0}(t, \tau, u_\tau)$. Thus, the graph is closed.

Also, there is a compact set K such that for all $\tau \in \mathbf{R}$ and $B \in \beta(H)$, there exists $T(\tau, B)$ such that $U_\Sigma(t, \tau, B) \subset K$ if $t \geq T$.

Indeed, for any $u(\cdot) \in D_{0,\sigma}(u_0)$ by (5.12), we obtain the inequality

$$\frac{d}{dt}\|u\|^2 + \widetilde{\alpha}\|u\|^2 \leq K_1\left(1 + \|h\|^2\right),$$

where $\widetilde{\alpha} > 0$. We note that

$$\begin{aligned}\int_0^t e^{\widetilde{\alpha s}}\|h(s)\|^2 ds &\leq \int_{t-1}^t e^{\widetilde{\alpha s}}\|h(s)\|^2 ds + \int_{t-2}^{t-1} e^{\widetilde{\alpha s}}\|h(s)\|^2 ds + \ldots \\ &\leq e^{\widetilde{\alpha t}}\int_{t-1}^t e^{\widetilde{\alpha s}}\|h(s)\|^2 ds + e^{\widetilde{\alpha(t-1)}}\int_{t-2}^{t-1}\|h(s)\|^2 ds + \ldots \\ &\leq \left(1 - e^{-\widetilde{\alpha}}\right)^{-1} e^{\widetilde{\alpha t}}\|h\|_b^2.\end{aligned}$$

Gronwall's lemma implies that

$$\|u(t)\|^2 \leq e^{-\widetilde{\alpha t}}\|u_0\|^2 + K_1\left(1 - e^{-\widetilde{\alpha}}\right)^{-1}\|h\|_b^2 + \frac{K_1}{\widetilde{\alpha}} = e^{-\widetilde{\alpha t}}\|u_0\|^2 + R^2, \text{ for all } t \geq 0. \tag{5.46}$$

Hence, the ball $B_0 = \left\{u \in H : \|u\| \leq \sqrt{R^2 + 1}\right\}$ is absorbing for the map $(t, u) \mapsto U_\Sigma(t, 0, u)$, that is, for any $B \in \beta(H)$, there exists $T(B)$ such that $U_\Sigma(t, 0, B) \subset B_0$, for $t \geq T$. In particular, U_σ is pointwise dissipative.

We define now the set $K = \overline{U_\Sigma(1, 0, B_0)}$. Lemma 5.2 implies that it is compact. Moreover, since B_0 is absorbing, using Lemma 5.9 for all $B \in \beta(H)$, we obtain the existence of $T(B, \tau)$ such that

$$\begin{aligned}U_\sigma(t, \tau, B) &= U_\sigma(t, t-1, U_\sigma(t-1, \tau, B)) \\ &= U_{T(t-1)\sigma}\left(1, 0, U_{T(\tau)\sigma}(t-1-\tau, 0, B)\right) \\ &\subset U_\Sigma(1, 0, B_0) \subset K,\end{aligned}$$

for all $\sigma \in \Sigma$, if $t \geq T(B, \tau)$.

It follows that any sequence $\{\xi_n\}$ such that $\xi_n \in U_{\sigma_n}(t_n, \tau, B)$, $\sigma_n \in \Sigma$, $t_n \to +\infty$, $B \in \beta(H)$, is precompact in H. Hence, the family U_σ is uniformly asymptotically upper semicompact. Also, by (5.46), we have $\gamma_{0,\Sigma}^\tau(B) \in \beta(H)$ for any $B \in \beta(H)$ and $\tau \in \mathbf{R}$.

As we have seen that $L1B - L3B$ hold, then all conditions of Theorem 4.5 and Lemma 4.8 hold. Thus, the existence of a uniform global compact invariant attractor follows.

We know by Theorem 5.3 that the maps U_σ have connected values. Also, it is clear that K is contained in some connected bounded set of H (e.g., a ball of sufficiently big radius). Therefore, the connectedness of the attractor follows from Theorem 4.6 if we check that Σ is a connected space. The map $s \mapsto T(s)\sigma$ is continuous for each $\sigma \in \Sigma$. Indeed, the continuity of $s \mapsto T(s) f$ in the space $C(\mathbf{R}^+, M)$ is evident from (5.44). The continuity of $s \mapsto T(s) h$ in the space $L_{2,w}^{loc}(\mathbf{R}, H)$ follows from the fact that $h_s(t) = h(t+s)$ converges to $h(t)$ in $L_2^{loc}(\mathbf{R}, H)$ as $s \to 0$ (see e.g., [6]). Hence, the set $\cup_{s\in\mathbf{R}}\sigma(\cdot + s)$ is connected, and so is the set Σ. □

If conditions (5.37)–(5.38) hold for any on any interval (τ, T), then Theorem 5.4 and Lemma 5.7 imply that nonnegative solutions exist for any $u_\tau \in H^+$, $\sigma \in \Sigma$, and then we can define also $D_{\tau,\sigma}^+(u_\tau) = \{u(\cdot) \in D_{\tau,\sigma}(u_\tau) : u(t) \in H^+$, for all $t \geq \tau\}$ and the map $U_\sigma^+ : \mathbf{R}_d^2 \times H^+ \to P(H^+)$ as

$$U_\sigma^+(t, \tau, u_\tau) = \left\{u(t) : u(\cdot) \in D_{\tau,\sigma}^+(u_\tau)\right\}.$$

As before, U_σ^+ is a multivalued strict dynamical process and $(Z3)$ holds.

In the same way as before, we prove the following:

Theorem 5.8. *Let (5.43)–(5.44) and (5.8)–(5.9) hold for $u \in \mathbf{R}_+^d$ (with constants not depending on $t \in \mathbf{R}$). Also, assume that (5.37)–(5.38) hold on any interval (τ, T). Then the MDP U_Σ^+ possesses the global compact invariant attractor Θ_Σ, which is connected in H^+.*

Remark 5.2. The results of Theorems 5.7, 5.8 remain valid if we change the Dirichlet boundary conditions by Neumann ones ($\frac{\partial u}{\partial \nu}|_{x\in\partial\Omega} = 0$).

Let us apply now these theorems to the complex Ginzburg–Landau equation and the Lotka–Volterra system.

5.3.1 Application to the Complex Ginzburg–Landau Equation

We consider first the Ginzburg–Landau equation (5.5). We assume now the following additional conditions:

$(GL1)$ $g^i \in L_2^{loc}(\mathbf{R}; L_2(\Omega))$ and (5.43) holds.
$(GL2)$ The continuous functions $R(t)$ and $\beta(t)$ satisfy

$$\begin{gathered} |\beta(s)| \leq D, |R(s)| \leq C, \\ |\beta(t) - \beta(s)| \leq a(|t-s|), \ \ |R(t) - R(s)| \leq b(|t-s|), \end{gathered} \tag{5.47}$$

for all $t, s \in \mathbf{R}$, where $a(l) \to 0$, $b(l) \to 0$, as $l \to 0^+$.

Condition (5.47) implies that (5.44) is satisfied. As a consequence of Theorem 5.7, we have:

Theorem 5.9. *Let $GL1 - GL2$ hold. Then (5.5) generates a family of MDP U_σ having a uniform global compact invariant attractor Θ_Σ, which is connected in $H = (L_2(\Omega))^2$.*

Proof. We have seen in Sect. 5.2.1 that (5.8)–(5.9) hold. Moreover, by (5.47), the constants do no depend on $t \in \mathbf{R}$. Also, $GL1 - GL2$ imply that (5.43)–(5.44) are satisfied. Then the result follows from Theorem 5.7. □

5.3.2 Application to the Lotka–Volterra System with Diffusion

Consider now the Lotka–Volterra system (5.6). As in the previous example, let us assume that the positive functions $a_i(t), a_{ij}(t)$ satisfy condition (5.47), so that (5.44) is satisfied. Also, we note that (5.37)–(5.38) hold on any interval (τ, T). As a consequence of Theorem 5.8, we have:

Theorem 5.10. *Let $a_i(t)$, $a_{ij}(t)$ satisfy conditions (5.47). Then (5.6) generates a family of MDP U_σ^+ having a uniform global compact invariant attractor Θ_Σ, which is connected in*

$$H^+ = \left\{ u \in (L_2(\Omega))^3 : u^i(x) \geq 0,\ \forall i, \text{for a.e. } x \in \Omega \right\}.$$

Proof. We have seen in Sect. 5.2.2 that (5.8)–(5.9) hold for $u \in \mathbf{R}_+^3$ with $p = (3, 3, 3)$, and by Lemma 5.7, we have that (5.37)–(5.38) are true. Moreover, by (5.47), the constants do no depend on $t \in \mathbf{R}$. Also, the new conditions on $a_i(t)$, $a_{ij}(t)$ imply that (5.43)–(5.44) hold. Hence, the result follows from Theorem 5.8 and Remark 5.2. □

Finally, let us consider the autonomous case, that is, let $f(u)$ and $h(x)$ do not depend on t.

In this case, we can define a strict multivalued semiflow $G : \mathbf{R}^+ \times H \to P(H)$ (i.e., $G(0, \cdot) = Id$, $G(t + s, x) = G(t, G(s, x))$, for all $t, s \geq 0, x \in H$) given by

$$G(t, u_0) = \{u(t) : u(\cdot) \in D_0(u_0)\},$$

where $D_0(u_0)$ is the set of all global weak solutions of problem (5.7) with $\tau = 0$. It is easy to see that considering that the set Σ contains a unique element σ, then $G(t, x) = U_\sigma(t, 0, x) = U_\Sigma(t, 0, x) = U_\Sigma(t + \tau, \tau, x)$, for any $\tau \in \mathbf{R}$. As a particular case of Theorem 5.7, or using the general theory of attractors for multivalued semiflows (see the proof of Theorem 10 in [13]), we obtain:

Theorem 5.11. *Let (5.8)–(5.9) hold. Then the multivalued semiflow G possesses the global compact invariant attractor Θ, which is connected in H.*

If (5.37)–(5.38) hold, then we can define also the map $G^+ : \mathbf{R}_+ \times H^+ \to P(H^+)$ as

$$G^{+}(t,u_0)=\left\{u(t):u(\cdot)\in D_0^{+}(u_\tau)\right\},$$

where $D_0^{+}(u_0)$ is the set of all global weak solutions of problem (5.7) with $\tau=0$ and such that $u(t)\in H^{+}$ for all $t\geq 0$. As before, we obtain:

Theorem 5.12. *Let (5.8)–(5.9) and (5.37)–(5.38) hold. Then the multivalued semiflow G^{+} possesses the global compact invariant attractor Θ, which is connected in H^{+}.*

Remark 5.3. As a corollary, we obtain that in the case where the parameters do not depend on t, (5.5) and system (5.6) generate the multivalued semiflows G and G^{+}. These semiflows have a global compact invariant attractor, which is connected in $H=(L_2(\Omega))^2$ and $H^{+}=\{u\in(L_2(\Omega))^3:u^i(x)\geq 0,\ \forall i$, for a.e. $x\in\Omega\}$, respectively.

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Chapter 6
Pullback Attractors for a Class of Extremal Solutions of the 3D Navier–Stokes System

The study of the asymptotic behavior of the weak solutions of the three-dimensional (3D for short) Navier–Stokes system is a challenging problem which is still far to be solved in a satisfactory way. In particular, the existence of a global attractor in the strong topology is an open problem for which only some partial or conditional results are given (see [3, 4, 6, 15, 17, 19, 20, 27, 38]). Concerning the existence of trajectory attractors, some results are proved in [13, 18, 36]. The main difficulty in this problem (but not the only one!) is to prove the asymptotic compactness of solutions (see [2], for a review on these questions).

With respect to the attractor in the weak topology, some results are proved in [15, 27]. Also, the Kneser's property (i.e., the compactness and connectedness of the attainability set for the weak solutions) in both the weak and strong topologies is studied in [28, 29].

In this chapter, we consider an optimal control problem associated with the 3D Navier–Stokes system which, in our point of view, could give some light on all these questions. Namely, let us consider the problem

$$\begin{cases} \frac{\partial y}{\partial t} - \nu \Delta y + (u \cdot \nabla) y = -\nabla p + f, \\ div\ y = 0, \\ y|_{\partial\Omega} = 0,\ y(\tau) = u_\tau, \end{cases} \tag{6.1}$$

where $\Omega \subset \mathbf{R}^3$ is a bounded open subset with smooth boundary, $\nu > 0$, and u is a control function belonging to a suitable set $\mathbf{U}_\tau$ (see Sect. 6.2). Then we will solve the following optimality problem: to find a pair $\{u, y\}$ such that y is a solution of (6.1) associated to u and an appropriate functional $J_\tau(u, y)$ attaches its infimum at (u, y).

We note first that this problem is nonautonomous. Also, we cannot guarantee uniqueness of the problem for a given initial data y_τ, although for a given u, the solution to (6.1) is unique. The reason is that more than one pair $\{u, y\}$ can exist as a solution of the optimality problem. Hence, in order to study the asymptotic behavior of solutions of such problem, we use the theory of pullback attractors for

M.Z. Zgurovsky et al., *Evolution Inclusions and Variation Inequalities for Earth Data Processing III*, Advances in Mechanics and Mathematics 27,
DOI 10.1007/978-3-642-28512-7_6,

multivalued processes developed in [9] (see also [1, 7, 8]). This theory generalizes the theory of pullback attractors for single-valued processes and cocycles, which has been studied intensively in the last years (see, e.g., [5, 10–12, 16, 31, 33, 35, 39], among many others).

We note that, comparing with the uniform attractor defined in Chap. 4, we use here a weaker concept of attractor, which allows to consider more general non-autonomous terms, not being in particular translation compact.

First, we develop the theory of pullback attractors for multivalued processes.

We prove secondly that the optimal problem has at least one solution for every initial data in the phase space H, and then, we construct a multivalued process associated to these solutions. The main theorem of the chapter states the existence of a strictly invariant pullback attractor for this process. Moreover, we prove the existence of a uniform global attractor for the process in the sense of [34] (see also [14] for the single-valued case), which contains the whole pullback attractor.

Finally, in the last section, we study the relationship of the pullback attractor of the optimal control problem with the global attractor of the 3D Navier–Stokes systems under the unproved condition that globally defined strong solutions exist for any initial data in V. In particular, we prove that in this case, the global attractor of the Navier–Stokes systems coincides with the pullback attractor. This result shows that there exists a close relation between the dynamics of the solutions of the optimal control problem and the dynamics of the solutions of the 3D Navier–Stokes system. Therefore, we hope that the optimal control problem will help us in the future to gain an insight into the problem on the existence of the global attractor for the 3D Navier–Stokes system.

These results were proved in [26].

6.1 Pullback Attractors for Multivalued Processes

Let X be a complete metric space with the metric ρ, $P(X)$ be the set of all nonempty subsets of X, and $\beta(X)$ be the set of all non-empty, bounded subsets of X. Put $R_d = \{(t,s) \in \mathbf{R}^2 : t \geq s\}$.

Definition 6.1. $U : R_d \times X \mapsto P(X)$ is called a multivalued process (m-process for short) if:

1. $U(\tau, \tau, x) = x, \quad \forall \tau \in \mathbf{R}, \forall x \in X.$
2. $U(t, \tau, x) \subseteq U(t, s, U(s, \tau, x)), \ \forall t \geq s \geq \tau, \ \forall x \in X.$

U is called strict if in (2) a strict equality holds.

For $t \in \mathbf{R}$, $B \in \beta(X)$, we define the omega-limit set

$$\omega(t, B) = \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} U(t, \tau, B)}.$$

Let $dist(A, B) = \sup_{y \in A} \inf_{x \in B} \rho(y, x)$. We shall denote by $O_\varepsilon(B) = \{y \in X : dist(y, B) < \varepsilon\}$ an ε-neighborhood of B.

Definition 6.2. The family of compact sets $\{\Theta(t)\}_{t \in \mathbf{R}}$ is called a pullback attractor if:

1. For any $t \in \mathbf{R}$, the set $\Theta(t)$ attracts every $B \in \beta(X)$ in the pullback sense, that is,
$$dist(U(t, \tau, B), \Theta(t)) \to 0, \text{ as } \tau \to -\infty. \tag{6.2}$$
2. $\Theta(t) \subseteq U(t, s, \Theta(s)), \forall\, t \geq s$ (negatively semiinvariance).
3. $\Theta(t)$ is the minimal closed pullback attracting set for all $t \in \mathbf{R}$.

The pullback attractor is strictly invariant if $\Theta(t) = U(t, s, \Theta(s)), \forall\, t \geq s$.

Theorem 6.1. *Let us suppose that there exists a family of compact sets $\{K(t)\}_{t \in \mathbf{R}}$ satisfying (6.2) and that the map $x \mapsto U(t, \tau, x)$ has closed graph for all $t \geq \tau$. Then there exists a pullback attractor $\{\Theta(t)\}_{t \in \mathbf{R}}$, $\Theta(t) \subset K(t), \forall t \in \mathbf{R}$, defined by*

$$\Theta(t) = \overline{\bigcup_{B \in \beta(X)} \omega(t, B)}.$$

Moreover, if there exists a closed set $B_0 \in \beta(X)$ such that for all $B \in \beta(X)$,

$$\sup_{\tau \in \mathbf{R}} dist\,(U(s + \tau, \tau, B), B_0) \to 0, \text{ as } s \to +\infty,$$

then $\Theta(t) = \omega(t, B_0) \subset B_0$.

In addition, if U is strict, then $\Theta(t) = U(t, s, \Theta(s))$, for any $t \geq s$, that is, $\Theta(t)$ is invariant.

Proof. We prove first the existence of a pullback attractor. First, we shall prove that for any $B \in \beta\,(X)$ and $t \in \mathbf{R}$, the ω-limit set $\omega\,(t, B) = \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} U\,(t, \tau, B)}$ is non-empty and compact and satisfies

$$dist\,(U\,(t, \tau, B)\,, \omega\,(t, B)) \to 0 \text{ as } \tau \to -\infty. \tag{6.3}$$

We note that $\xi \in \omega\,(t, B)$ if and only if there exist sequences $\xi_n \in U\,(t, \tau_n, B)$, with $\tau_n \to -\infty$, such that $\xi_n \to \xi$. We take an arbitrary sequence $\xi_n \in U\,(t, \tau_n, B)$, where $\tau_n \to -\infty$. In view of

$$dist\,(U\,(t, \tau, B)\,, K\,(t)) \to 0, \text{ as } \tau \to -\infty, \tag{6.4}$$

and the compactness of $K\,(t)$, it is easy to see that ξ_n has a converging subsequence ξ_{n_k}. Hence, $\xi_{n_k} \to \xi \in \omega\,(t, B)$, so that $\omega\,(t, B)$ is nonempty.

If $\xi \in \omega(t, B)$, we have that $\xi_n \to \xi$, where $\xi_n \in U\,(t, \tau_n, B)$ and $\tau_n \to -\infty$. Thus, $\xi \in K\,(t)$ and then $\omega\,(t, B) \subset K\,(t)$, which implies that it is compact.

Suppose further that (6.3) is not true. Then we can find $\varepsilon > 0$ and a sequence $\xi_n \in U(t, s_n, B)$, where $s_n \to -\infty$, such that $dist(\xi_n, \omega(t, B)) > \varepsilon$, for all n. But condition (6.4) implies that $\{\xi_n\}$ has a converging subsequence ξ_{n_k} and that $\xi_{n_k} \to \xi \in \omega(t, B)$, which is a contradiction.

Also, $\omega(t, B)$ is the minimal closed set satisfying (6.3). Indeed, let Y be a closed set for which (6.3) holds. For any $y \in \omega(t, B)$, we can obtain a sequence $\xi_n \in U(t, s_n, B)$ converging to y as $s_n \to -\infty$. Take an arbitrary $\varepsilon > 0$. Since Y satisfies (6.3), there exists n_0 such that $dist(\xi_n, Y) < \frac{\varepsilon}{2}$ and $\rho(y, \xi_n) < \frac{\varepsilon}{2}$, for all $n > n_0$. Therefore,

$$dist(y, Y) \le \rho(y, \xi_n) + dist(\xi_n, Y) < \varepsilon.$$

Since Y is closed, we finally obtain that $y \in Y$.

Moreover, $\omega(t, B)$ is semiinvariant, that is, $\omega(t, B) \subset U(t, s, B(s, B))$ for all $t \ge \tau$. Indeed, for an arbitrary $y \in \omega(t, B)$ we can find a sequence $\xi_n \in U(t, s_n, B)$ converging to y as $s_n \to -\infty$. For any $s_n \le \tau \le t$, we have

$$U(t, s_n, B) \subset U(t, \tau, U(\tau, s_n, B)),$$

so that $\xi_n \in U(t, \tau, \zeta_n)$, where $\zeta_n \in U(\tau, s_n, B)$. It follows again by (6.4) that passing to a subsequence $\zeta_n \to \zeta \in \omega(\tau, B)$. Since the graph of $x \mapsto U(t, \tau, x)$ is closed, we get $y \in U(t, \tau, \zeta) \subset U(t, \tau, \omega(\tau, B))$. It follows that

$$\omega(t, B) \subset U(t, \tau, \omega(\tau, B)),$$

for any $B \in \beta(X)$, and then, $\omega(t, B)$ is semiinvariant.

It follows from (6.3) that $\Theta(t)$ satisfies the first property in Definition 6.2. Also, the minimality property is a consequence of the minimality of the omega-limit sets $\omega(t, B)$. It remains to prove that $\Theta(t) \subset U(t, \tau, \Theta(\tau))$ for $t \ge \tau$. Let $y \in \Theta(t)$ be arbitrary. Then there exists a sequence $y_n \in \omega(t, B_n)$, $B_n \in \beta(X)$, converging to y. Since the omega-limit sets are semiinvariant, for any $\tau < t$, we can obtain a sequence $\zeta_n \in \omega(\tau, B_n)$ such that $y_n \in U(t, \tau, \zeta_n)$. By the compactness of $K(\tau)$, we can assume that $\zeta_n \to \zeta \in \Theta(\tau)$. Finally, using the fact that the map $X \ni x \longmapsto U(t, \tau, x)$ has closed graph, we have $y \in U(t, \tau, \zeta) \subset U(t, \tau, \Theta(\tau))$. Hence, $\Theta(t) \subset U(t, \tau, \Theta(\tau))$.

Hence, we have proved that $\Theta(t)$ is a pullback attractor.

Further, we can write

$$\omega(t, B) = \bigcap_{T \ge 0} \overline{\bigcup_{s \ge T} U(t, t-s, B)}.$$

So, putting $\tau_s = t - s$, from

$$\bigcup_{s \ge T} U(t, t-s, B) = \bigcup_{s \ge T} U(\tau_s + s, \tau_s, B) \subset O_\varepsilon(B_0), \ \forall T \ge T(B, \varepsilon),$$

for an arbitrary small $\varepsilon > 0$, it follows that $\omega(t, B) \subset B_0$ and, hence, $\Theta(t) \subset B_0$.

On the other hand,

$$\omega(t, B) \subset U(t, s, \omega(s, B)) \subset U(t, s, B_0) \to \omega(t, B_0), \text{ as } s \to -\infty.$$

So, $\omega(t, B) \subset \omega(t, B_0)$, and it follows that $\Theta(t) = \omega(t, B_0)$.

If U is strict, then from

$$\omega(t, B_0) \subset U(t, s, \omega(s, B_0)), \ \forall t \geq s,$$

we obtain

$$U(p, t, \omega(t, B_0)) \subset U\big(p, t, U(t, s, \omega(s, B_0))\big) \subset U(p, s, B_0).$$

Since $U(p, s, B_0) \to \omega(p, B_0)$, as $s \to -\infty$, we obtain

$$U(p, t, \omega(t, B_0)) \subset \omega(p, B_0), \forall p \geq t,$$

and then,

$$\Theta(p) = U(p, t, \Theta(t)), \forall p \geq t.$$

□

6.2 Setting of the Problem and Main Results

Let $\Omega \subset \mathbf{R}^3$ be a bounded open subset with smooth boundary. We shall define the usual function spaces

$$\mathscr{V} = \{u \in (C_0^\infty(\Omega))^3 : div\, u = 0\},$$
$$H = cl_{(L_2(\Omega))^3}\mathscr{V}, \ V = cl_{(H_0^1(\Omega))^3}\mathscr{V},$$

where cl_X denotes the closure in the space X. It is well known that H, V are separable Hilbert spaces, and identifying H and its dual H^*, we have $V \subset H \subset V^*$ with dense and continuous injections. We denote by $(\cdot,\cdot)$, $|\cdot|$ and $((\cdot,\cdot))$, $\|\cdot\|$ the inner product and norm in H and V, respectively. $\langle\cdot,\cdot\rangle$ will denote pairing between V and V^*. We set $\mathbf{L}_4(\Omega) = (L_4(\Omega))^3$ with the norm denoted by $\|\cdot\|_{\mathbf{L}_4}$. We will denote by B_R a closed ball with radius R and centered at 0 in the space H.

For $u, v, w \in V$, we put

$$b(u, v, w) = \int\limits_\Omega \sum_{i,j=1}^3 u_i \frac{\partial v_j}{\partial x_i} w_j \, dx.$$

It is known [37] that b is a trilinear continuous form on V and $b(u, v, v) = 0$, if $u \in V, v \in \left(H_0^1(\Omega)\right)^3$. As usual, for $u, v \in V$, we denote by $B(u, v)$ the element of V^* defined by $\langle B(u, v), w\rangle = b(u, v, w)$, for all $w \in V$.

We consider also the Stokes operator $A : D(A) \to H$, $D(A) = \left(H^2(\Omega)\right)^3 \cap V$, where $Au = -P\Delta u$, P is the Helmholtz–Leray projector and Δ is the Laplacian operator (see, e.g., [20] for more details).

We consider the 3D controlled Navier–Stokes system

$$\begin{cases} \frac{dy}{dt} + Ay + B(u, y) = f, \\ y(\tau) = y_\tau \in H, \end{cases} \tag{6.5}$$

where $f \in H$ and

$$u(\cdot) \in \mathbf{U}_\tau = \begin{cases} u \in L_\infty(\tau, +\infty; H) \cap L_2^{loc}(\tau, +\infty; V) \cap L_\infty^{loc}(\tau, +\infty; \mathbf{L}_4(\Omega)), \\ \int\limits_\tau^{+\infty} \|u(p)\|^2 e^{-\delta p} dp < \infty, \ |u(p)| \le R_0 \text{ for a.a. } p \ge \tau, \\ \|u(t)\|_{\mathbf{L}_4} \le \alpha \text{ for a.a. } t > \tau, \end{cases} \tag{6.6}$$

$$J_\tau(u, y) = \int\limits_\tau^{+\infty} \|y(p) - u(p)\|^2 e^{-\delta p} dp \to \inf, \tag{6.7}$$

with $\delta = \lambda_1 \nu$, $R_0 = \frac{|f|}{\nu\lambda_1}$ and where λ_1 is the first eigenvalue of the Stokes operator A and $\alpha > 0$ is some constant.

We have two aims. The first one is to prove that the solutions of the optimal control problem (6.5)–(6.7) generate a multivalued process U, which has a pullback attractor. The second aim is to prove that under the unproved assumption about strong global solvability of the 3D Navier–Stokes system, the pullback attractor coincides with the global attractor of the multivalued semiflow for the 3D Navier–Stokes system given in [27].

By using standard Galerkin approximations (see [37]), it is easy to show that for any $y_\tau \in H$ and $u(\cdot) \in U_\tau$, there exists a unique weak solution $y(\cdot) \in L_\infty(\tau, +\infty; H) \cap L_2^{loc}(\tau, +\infty; V)$ of (6.5), that is,

$$\frac{d}{dt}(y, v) + \nu((y, v)) + b(u, y, v) = \langle f, v\rangle, \text{ for all } v \in V. \tag{6.8}$$

Indeed, let us prove existence of a weak solution of (6.5). Let $\{w_i\} \subset D(A)$ be the sequence of eigenfunctions of A, which are an orthonormal basis of H. Let $y^m(t) = \sum_{i=1}^m g_{im}(t) w_i$ be the Galerkin approximations of (6.5), i.e.

$$\begin{cases} \frac{dy^m}{dt} + \nu Ay^m + P_m B(u, y^m) = P_m f, \\ y^m(\tau) = y_\tau^m, \end{cases} \tag{6.9}$$

where P_m is the projection onto the finite-dimensional subspace generated by the set $\{w_1, \ldots, w_m\}$. Also, y_τ^m belongs to this subspace and $y_\tau^m \to y_\tau$ in H.

We need to obtain some a priori estimates for the approximative functions $\{y^m\}$. Multiplying (6.9) by y^m, we obtain

$$\frac{1}{2}\frac{d}{dt}|y^m|^2 + \nu \|y^m\|^2 = (f, y^m), \tag{6.10}$$

where we have used the equalities $(P_m B(u, y^m), y^m) = (B(u, y^m), y^m) = b(u, y^m, y^m) = 0$.

Also from (6.10), we obtain for all $p \in [s, T]$, $s \in [\tau, T]$ that

$$\frac{1}{2}|y^m(p)|^2 + \nu \int_s^p \|y^m(\tau)\|^2 d\tau \leq \int_s^p (f(\tau), y^m(\tau))\, d\tau + \frac{1}{2}|y^m(s)|^2. \tag{6.11}$$

In view of (6.11), we conclude that $\{y^m\}$ is bounded in $L_2(\tau, T; V) \cap L_\infty(\tau, T; H)$.

Therefore, passing to a subsequence, we obtain $y^m \to y$ weakly in $L_2(\tau, T; V)$ and weakly star in $L_\infty(\tau, T; H)$. From the inequalities

$$|b(u, y^m, w)| \leq d\|u\|_{L_4}\|y^m\|\|w\|, \ \forall w \in V,$$

and

$$\|P_m B(u, y^m)\|_{V^*} \leq \|B(u, y^m)\|_{V^*},$$

due to the choice of the spacial basis, we immediately obtain that $P_m B(u, y^m)$ is bounded in $L_2(\tau, T; V^*)$. Then

$$\frac{d}{dt}y^m \to \frac{d}{dt}y \text{ weakly in } L_2\left(\tau, T; V^*\right),$$

so that $y \in C([\tau, T]; H)$, and by the Compactness Lemma (see Theorem 5.2), we have

$$y^m \to y \text{ strongly in } L_2(\tau, T; H).$$

Hence, $y^m(t) \to y(t)$ strongly in H for a.a $t \in (\tau, T)$. Since one can easily prove using the Ascoli–Arzelà theorem that $y^m \to y$ in $C([\tau, T]; V^*)$, a standard argument implies that $y^m(t) \to y(t)$ weakly in H for all $t \in [\tau, T]$. In particular, $y(\tau) = y_\tau$.

On the other hand, from

$$\left\|u_i y_j^m\right\|^2_{L_2(\tau,T;L_2(\Omega))} \leq \int_\tau^T \|u_i\|^2_{L_4(\Omega)} \left\|y_j^m\right\|^2_{L_4(\Omega)} dt \leq C,$$

we obtain $u_i y_j^m \to u_i y_j$ weakly in $L_2(\tau, T; L_2(\Omega))$, so that

$$\int_{\tau}^{T} b(u, y^m - y, w)dt = -\sum_{i,j=1}^{3} \int_{\tau}^{T} \int_{\Omega} u_i \left(y_j^m - y_j\right) \frac{\partial w_j}{\partial x_i} dxdt \to 0,$$

for any $w \in L_2(\tau, T; V)$. This implies

$$B(u, y^m) \to B(u, y) \text{ weakly in } L_2\left(\tau, T; V^*\right).$$

So we can pass to the limit in (6.9) and deduce that y is solution of (6.5). To prove uniqueness, we should note that if y_1, y_2 are solutions of (6.5), corresponding the same control function u, then

$$\frac{d}{dt}|y_1 - y_2|^2 = 2(\frac{d(y_1 - y_2)}{dt}, y_1 - y_2),$$
$$b(u, y_1 - y_2, y_1 - y_2) = 0.$$

So after simple calculations, we have

$$\frac{d}{dt}|y_1 - y_2|^2 \leq C|y_1 - y_2|^2,$$

and therefore, $y_1 \equiv y_2$.

Moreover, by the inequality

$$|b(u, y, v)| = |b(u, v, y)| \leq c_1 \|u\|_{\mathbf{L}_4} \|v\| \|y\|_{\mathbf{L}_4} \leq c_2 c_1 \|u\|_{\mathbf{L}_4} \|v\| \|y\|, \forall u, y, v \in V,$$

and (6.6), we have $B(u(\cdot), y(\cdot)) \in L_2^{loc}(\tau, +\infty; V^*)$, so $\frac{dy}{dt} \in L_2^{loc}(\tau, +\infty; V^*)$ as well. Hence, it follows that $y(\cdot) \in C([\tau, +\infty); H)$ (so the initial condition $y(\tau) = y_\tau$ makes sense for any $y_\tau \in H$) and standard arguments imply that for all $t \geq s \geq \tau$,

$$F(y(t)) := \left(|y(t)|^2 - R_0^2\right)e^{\delta t} \leq F(y(s)), \tag{6.12}$$

$$V_\tau(y(t)) := \frac{1}{2}|y(t)|^2 + \nu \int_{\tau}^{t} \|y(p)\|^2 dp - \int_{\tau}^{t} (f, y(p))dp \leq V_\tau(y(s)), \tag{6.13}$$

$$|y(t)|^2 + \nu \int_{\tau}^{t} \|y(p)\|^2 dp \leq |y_\tau|^2 + \frac{|f|^2}{\nu\lambda_1}(t - \tau). \tag{6.14}$$

Indeed, multiplying the equation by $y(t)$ and using the property $b(u, y, y) = 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}|y|^2 + \nu\|y\|^2 = (f, y). \tag{6.15}$$

After integration over (s, t), we obtain

$$\frac{1}{2}|y(t)|^2 + \nu \int_s^t \|y(p)\|^2\, dp = \int_s^t (f, y(p))\, dp + \frac{1}{2}|y(s)|^2, \tag{6.16}$$

and then (6.13) follows. Taking $s = \tau$ in (6.16) and using the inequality

$$(f, y(p)) \le |f|\,|y(p)| \le \frac{1}{\sqrt{\lambda_1}}|f|\,\|y(p)\| \le \frac{|f|^2}{2\lambda_1 \nu} + \frac{\nu}{2}\|y(p)\|^2,$$

we have

$$|y(t)|^2 + \nu \int_s^t \|y(p)\|^2\, dp \le |y(\tau)|^2 + \frac{|f|^2}{\lambda_1 \nu}(t - \tau).$$

Finally, from (6.15), we obtain

$$\frac{d}{dt}|y|^2 + \lambda_1 \nu |y|^2 \le \frac{|f|^2}{\lambda_1 \nu}.$$

Multiplying the last inequality by $e^{\nu\lambda_1 t}$ and integrating, we get

$$|y(t)|^2 e^{\nu\lambda_1 t} \le |y(s)|^2 e^{\nu\lambda_1 s} + \frac{|f|^2}{\lambda_1^2 \nu^2}(e^{\lambda_1 \nu t} - e^{\nu\lambda_1 s}),$$

and then, (6.12) holds.

So, for all $n \ge 0$,

$$\begin{aligned}
\int_{\tau+n}^{\tau+(n+1)} \|y(p) - u(p)\|^2 e^{-\delta p}\, dp &\le 2e^{-\delta(n+\tau)} \int_{\tau+n}^{\tau+(n+1)} \|y(p)\|^2 dp \\
&\quad + 2 \int_{\tau+n}^{\tau+(n+1)} \|u(p)\|^2 e^{-\delta p}\, dp \\
&\le \frac{2}{\nu} e^{-\delta(n+\tau)} \left(|y_\tau|^2 + \frac{|f|^2}{\nu\lambda_1} \right) \\
&\quad + 2 \int_{\tau+n}^{\tau+(n+1)} \|u(p)\|^2 e^{-\delta p}\, dp.
\end{aligned}$$

From this,

$$J_\tau(u,y) = \sum_{n=0}^{\infty} \int_{\tau+n}^{\tau+(n+1)} \|y(p) - u(p)\|^2 e^{-\delta p} dp$$

$$\le \frac{2e^{-\delta\tau}}{\nu}(|y_\tau|^2 + \frac{|f|^2}{\nu\lambda_1}) \sum_{n=0}^{\infty} e^{-\delta n} + 2\sum_{n=0}^{\infty} \int_{\tau+n}^{\tau+(n+1)} \|u(p)\|^2 e^{-\delta p} dp < \infty.$$

Therefore, the functional J_τ and the optimal control problem (6.5)–(6.7) are correctly defined.

Lemma 6.1. *For any $\tau \in \mathbf{R}$ and $y_\tau \in H$, the optimal control problem (6.5)–(6.7) has at least one solution $\{y(\cdot), u(\cdot)\}$, and, moreover, $\frac{dy}{dt} \in L_2^{loc}(\tau, +\infty; V^*)$, $y(\cdot) \in C([\tau, +\infty); H)$ and (6.12)–(6.14) hold.*

Proof. Let $\{y_n, u_n\}$ be a minimizing sequence such that

$$\int_\tau^{+\infty} \|y_n(p) - u_n(p)\|^2 e^{-\delta p} dp \le d + \frac{1}{n},\ \forall n \ge 1\ ,$$

where $d = \inf J_\tau(u,y)$. Thus, for all $T > \tau$,

$$\begin{aligned} &\int_\tau^T \|y_n(p) - u_n(p)\|^2 e^{-\delta p} dp \le d + \tfrac{1}{n}, \\ &\int_\tau^T \|y_n(p) - u_n(p)\|^2 dp \le (d + \tfrac{1}{n}) e^{\delta T}. \end{aligned} \tag{6.17}$$

From (6.12) to (6.14), we obtain that $\{y_n\}$ is bounded in $L_\infty(\tau,T;H) \cap L_2(\tau,T;V)$. Hence, (6.17) implies that $\{u_n\}$ is bounded in $L_2(\tau,T;V)$ and from the definition of $\mathbf{U}_\tau$, it follows that

$$\begin{aligned} &|u_n(p)| \le R_0,\ \forall p \ge \tau, \\ &\|u_n(p)\|_{\mathbf{L}^4} \le \alpha \text{ for a.a. } p > \tau. \end{aligned}$$

Therefore, there exist $u \in L_\infty(\tau,T;H) \cap L_2(\tau,T;V) \cap L_\infty(\tau,T;\mathbf{L}_4(\Omega))$ and $y \in L_\infty(\tau,T;H) \cap L_2(\tau,T;V)$ such that

$$\begin{aligned} &u_n \to u \text{ weakly in } L_2(\tau,T;V), \\ &u_n \to u\ * -\text{weakly in } L_\infty(\tau,T;H), \\ &u_n \to u\ * -\text{weakly in } L_\infty(\tau,T;\mathbf{L}^4(\Omega)), \\ &y_n \to y \text{ weakly in } L_2(\tau,T;V), \\ &y_n \to y\ * -\text{weakly in } L_\infty(\tau,T;H). \end{aligned} \tag{6.18}$$

Moreover, $\|B(u_n,y_n)\|_{V^*} \le c_1\|y_n\|\|u_n\|_{\mathbf{L}_4}$. Hence, $\frac{dy_n}{dt}$ is bounded in $L_2(\tau,T;V^*)$. From this using standard arguments, we obtain that $y(\cdot) \in C([\tau,T];H)$ is the solution of (6.5) with control $u(\cdot)$, $y(\cdot)$ satisfies (6.12)–(6.14), and for this control, the following relations hold:

$$|u(p)| \leq R_0, \quad \text{for a.a. } p \geq \tau,$$
$$\|u(p)\|_{\mathbf{L}_4} \leq \alpha \quad \text{for a.a.} \quad p > \tau,$$
$$u \in L_2(\tau, T; V),$$
$$\int_\tau^T \|y(p) - u(p)\|^2 e^{-\delta p} dp \leq d.$$

The fact that $y(\cdot)$ is a solution with control $u(\cdot)$ is proved in a standard way. Indeed, as $\frac{dy_n}{dt}$ is bounded in $L_2(\tau, T; V^*)$, up to subsequence

$$\frac{d}{dt} y_n \to \frac{d}{dt} y \text{ weakly in } L_2\left(\tau, T; V^*\right).$$

Thus, $y \in C([\tau, T]; H)$ and arguing as in the proof of the existence of solution for (6.5), we obtain

$$y_n \to y \text{ strongly in } L_2(\tau, T; H),$$
$$y_n(t) \to y(t) \text{ strongly in } H \text{ for a.a } t \in (\tau, T),$$
$$y_n(t) \to y(t) \text{ weakly in } H \text{ for all } t \in [\tau, T].$$

From

$$\left\| u_i^n y_j^n \right\|^2_{L_2(\tau,T;L_2(\Omega))} \leq \int_\tau^T \|u_i^n\|^2_{L_4(\Omega)} \left\| y_j^n \right\|^2_{L_4(\Omega)} dt \leq C,$$

we obtain $u_i y_j^m \to u_i y_j$ weakly in $L_2(\tau, T; L_2(\Omega))$, so that

$$\int_\tau^T b(u^n, y^n, w) dt = -\sum_{i,j=1}^{3} \int_\tau^T \int_\Omega u_i^n y_j^n \frac{\partial w_j}{\partial x_i} dx dt \to \int_\tau^T b(u, w, y) dt,$$

for any $w \in L_2(\tau, T; V)$. This implies

$$B(u, y^m) \to B(u, y) \text{ weakly in } L_2\left(\tau, T; V^*\right).$$

Hence, we can pass to the limit in (6.5) and obtain that $\{u, y\}$ is a solution. Also, $y(\tau) = y_\tau$.

By using a standard diagonal procedure, we can claim that $y(\cdot)$ and $u(\cdot)$ are defined on $[\tau, +\infty)$, $y_n \to y$, $u_n \to u$ in the previous sense on every $[\tau, T]$, and

$$\int_\tau^{+\infty} \|y(p) - u(p)\|^2 e^{-\delta p} dp \leq d. \tag{6.19}$$

By (6.14), arguing as before,

$$\int_{\tau}^{\infty} \|y(p)\|^2 e^{-\delta p} dp = \sum_{n=0}^{\infty} \int_{\tau+n}^{\tau+n+1} \|y(p)\|^2 e^{-\delta p} dp$$
$$\leq \frac{e^{-\delta\tau}}{\nu} \left(|y_\tau|^2 + \frac{|f|^2}{\nu\lambda_1} \right) \sum_{n=0}^{\infty} e^{-\delta n} < \infty,$$

and from (6.19), we have

$$\int_{\tau}^{+\infty} \|u(p)\|^2 e^{-\delta p} dp < \infty.$$

It follows that $u(\cdot) \in \mathbf{U}_\tau$ and from (6.19), we obtain that $\{y(\cdot), u(\cdot)\}$ is an optimal pair of problem (6.5)–(6.7). □

Now, we are ready to construct a multivalued process associated to the solutions of problem (6.5)–(6.7).

For every $y_\tau \in H$, $\tau \in \mathbf{R}$, and $t \geq \tau$, we put

$$U(t,\tau,y_\tau) = \left\{ \tilde{y}(t) \;\middle|\; \begin{array}{l} \tilde{y}(\cdot) \text{ is a solution of the} \\ \text{optimal problem (6.5)–(6.7)}, \; \tilde{y}(\tau) = y_\tau \end{array} \right\}. \tag{6.20}$$

Lemma 6.2. *The multivalued map U defined by (6.20) is a strict multivalued process.*

Proof. It is obvious that $U(\tau,\tau,y_\tau) = y_\tau$.

Let $\xi \in U(t,\tau,y_\tau)$. Thus, $\xi = \tilde{y}(t)$, where $\{\tilde{y}(\cdot), \tilde{u}(\cdot)\}$ is an optimal pair of problem (6.5)–(6.7) with $\tilde{y}(\tau) = y_\tau$, $\tilde{u}(\cdot) \in \mathbf{U}_\tau$. Then of course, $\tilde{y}(s) \in U(s,\tau,y_\tau)$, for all $s \in (\tau,t)$. We should prove that $\tilde{y}(t) \in U(t,s,\tilde{y}(s))$, that is, Bellman's principle of optimality. Let

$$\tilde{J}_\tau = J_\tau(\tilde{y},\tilde{u}) = \int_{\tau}^{+\infty} \|\tilde{y}-\tilde{u}\|^2 e^{-\delta p} dp = \int_{\tau}^{s} \|\tilde{y}-\tilde{u}\|^2 e^{-\delta p} dp + \int_{s}^{+\infty} \|\tilde{y}-\tilde{u}\|^2 e^{-\delta p} dp.$$

We consider the problem (6.5)–(6.7) on the interval $[s,+\infty)$ (formally s instead of τ) with the set of controls $\mathbf{U}_s$ and the initial data $(s,\tilde{y}(s))$. It is easy to verify that $\{\tilde{y}(\cdot), \tilde{u}(\cdot)\}$ on $[s,+\infty)$ is a solution of the optimal control problem. Indeed, we note that $\tilde{u}(\cdot) \in \mathbf{U}_s$ and that $\tilde{y}(\cdot)$ is the unique solution of (6.5) corresponding to $\tilde{u}(\cdot)$. Let $\{\hat{y}(\cdot), \hat{u}(\cdot)\}$ be an optimal pair of this problem. Suppose that

$$\int_{s}^{+\infty} \|\hat{y}-\hat{u}\|^2 e^{-\delta p} dp < \int_{s}^{+\infty} \|\tilde{y}-\tilde{u}\|^2 e^{-\delta p} dp.$$

Let us consider the control

$$u(t) = \begin{cases} \tilde{u}(t), \ t \in [\tau, s), \\ \hat{u}(t), \ t \in [s, +\infty). \end{cases}$$

We claim that $u(\cdot) \in \mathbf{U}_\tau$. Indeed, it is clear that $|u(p)| \leq R_0$ for a.a. $p \geq \tau$, $\|u(t)\|_{\mathbf{L}^4} \leq \alpha$ for a.a. $t > \tau$,

$$u(\cdot) \in L_\infty(\tau, +\infty; H) \cap L_2^{loc}(\tau, +\infty; V),$$

$$\int_\tau^{+\infty} \|u(p)\|^2 e^{-\delta p} dp < \infty.$$

Then $u(\cdot) \in \mathbf{U}_\tau$ and

$$y(t) = \begin{cases} \tilde{y}(t), \ t \in [\tau, s), \\ \hat{y}(t), \ t \in [s, +\infty), \end{cases}$$

is the solution of problem (6.5) which corresponds to the control $u(\cdot)$ (because of uniqueness of the solution of problem (6.5) for a fixed $u(\cdot)$).

Finally,

$$J_\tau(u, y) = \int_\tau^s \|\tilde{y} - \tilde{u}\|^2 e^{-\delta p} dp + \int_s^{+\infty} \|\hat{y} - \hat{u}\|^2 e^{-\delta p} dp \geq \tilde{J}_\tau =$$

$$= \int_\tau^s \|\tilde{y} - \tilde{u}\|^2 e^{-\delta p} dp + \int_s^{+\infty} \|\tilde{y} - \tilde{u}\|^2 e^{-\delta p} dp,$$

which is a contradiction. Hence, $\{\tilde{y}(\cdot), \tilde{u}(\cdot)\}$ is an optimal pair on $[s, +\infty)$ and then $\tilde{y}(t) \in U(t, s, \tilde{y}(s))$, so that U is a multivalued process.

Let us prove that it is strict. Let $\xi \in U(t, s, U(s, \tau, y_\tau))$. Then $\xi = \tilde{y}_2(t)$ and $\{\tilde{y}_2(\cdot), \tilde{u}_2(\cdot)\}$ is an optimal pair of problem (6.5)–(6.7) with $\tilde{y}_2(s) = y_s$, $\tilde{u}_2(\cdot) \in \mathbf{U}_s$, and $y_s = \tilde{y}_1(s)$, where $\{\tilde{y}_1(\cdot), \tilde{u}_1(\cdot)\}$ is an optimal pair of problem (6.5)–(6.7) with $\tilde{y}_1(\tau) = y_\tau$, $\tilde{u}_1(\cdot) \in \mathbf{U}_\tau$. Let us consider the control

$$u(t) = \begin{cases} \tilde{u}_1(t), \ t \in [\tau, s), \\ \tilde{u}_2(t), \ t \in [s, +\infty). \end{cases}$$

As before, $u \in \mathbf{U}_\tau$ and

$$y(t) = \begin{cases} \tilde{y}_1(t), \ t \in [\tau, s), \\ \tilde{y}_2(t), \ t \in [s, +\infty) \end{cases}$$

are the solutions of problem (6.5) which correspond to the control $u(\cdot)$. Also, it is clear, as $\tilde{y}_1(\cdot)$ is a solution of problem (6.5) on $[s, +\infty)$ corresponding to $\tilde{u}_1(\cdot)$ that

$$J_\tau(u, y) = \int_\tau^s \|\tilde{y}_1 - \tilde{u}_1\|^2 e^{-\delta p} dp \; + \int_s^{+\infty} \|\tilde{y}_2 - \tilde{u}_2\|^2 e^{-\delta p} dp$$

$$\leq \int_\tau^s \|\tilde{y}_1 - \tilde{u}_1\|^2 e^{-\delta p} dp \; + \int_s^{+\infty} \|\tilde{y}_1 - \tilde{u}_1\|^2 e^{-\delta p} dp.$$

Hence, $\xi = y(t) \in U(t, \tau, y_\tau)$. □

Theorem 6.2. *For the multivalued process U, given by (6.20), there exists a strictly invariant pullback attractor $\{\Theta(t)\}_{t\in\mathbf{R}}$ such that $\Theta(t) \subset B_{R_0}$, for all $t \in \mathbf{R}$.*

Proof. First of all, from (6.12) for every $R > R_0$, $y_\tau \in H$ such that $|y_\tau| \leq R$, it holds

$$|y(s+\tau)|^2 - R_0^2 \leq e^{-\delta(s+\tau)}\big(|y_\tau|^2 - R_0^2\big)e^{\delta\tau},$$

$$|y(s+\tau)|^2 \leq e^{-\delta s}(R^2 - R_0^2) \; + \; R_0^2.$$

So,

$$\sup_{\tau\in\mathbf{R}} dist\big(U(s+\tau, \tau, B_R), B_{R_0}\big) \to 0, \text{ as } s \to +\infty. \tag{6.21}$$

In virtue of Theorem 6.1 and

$$U(t, s, B_R) \subset U(t, t-1, U(t-1, s, B_R)) \subset U(t, t-1, B_{R_0+1}),$$

where the last inclusion follows from (6.21) by taking a sufficiently small s, we only need to prove that the set $K(t) := \overline{U(t, t-1, B_{R_0+1})}$ is compact and that the map $x \mapsto U(t, \tau, x)$ has closed graph. These two properties are true, if the following statements holds for all $t \geq \tau$:

- $(U1)$ If $\eta_n \to \eta$ weakly in H and $\xi_n \in U(t, \tau, \eta_n)$, then the sequence $\{\xi_n\}$ is precompact in H.
- $(U2)$ If $\eta_n \to \eta$ strongly in H and $\xi_n \in U(t, \tau, \eta_n)$, then up to subsequence $\xi_n \to \xi \in U(t, \tau, \eta)$.

Let $\xi_n \in U(t, \tau, \eta_n)$, where $\eta_n \to \eta$ weakly in H. Then $\xi_n = \tilde{y}_n(t)$, $\tilde{y}_n(\tau) = \eta_n$, where $\{\tilde{y}_n(\cdot), \tilde{u}_n(\cdot)\}$ is an optimal pair of problem (6.5)–(6.7), $\tilde{u}_n(\cdot) \in \mathbf{U}_\tau$. We have that $\{\tilde{y}_n(\cdot)\}$ satisfy (6.12)–(6.14). If we consider the control $u(\cdot) \equiv 0 \in \mathbf{U}_\tau$ and the corresponding solution of (6.5), $y_n(\cdot)$, with $y_n(\tau) = \eta_n$, then

$$J_\tau(\tilde{u}_n, \tilde{y}_n) \leq J_\tau(0, y_n) = \int_\tau^{+\infty} \|y_n(p)\|^2 e^{-\delta p} dp = \sum_{k=0}^{\infty} \int_{\tau+k}^{\tau+k+1} \|y_n(p)\|^2 e^{-\delta p} dp.$$

But $y_n(\cdot)$ satisfies (6.12)–(6.14), so

$$\int_{\tau+k}^{\tau+k+1} \|y_n(p)\|^2 e^{-\delta p} dp \le \frac{e^{-\delta(k+\tau)}}{\nu}(|\eta_n|^2 + \frac{|f|^2}{\nu\lambda_1}).$$

Thus, $J_\tau(\tilde{u}_n, \tilde{y}_n) \le \tilde{C}$, where $\tilde{C}$ does not depend on n. Then in the same way as in Lemma 6.1, we obtain (up to a subsequence) that $\tilde{u}_n \to \tilde{u} \in \mathbf{U}_\tau$, $\tilde{y}_n \to \tilde{y}$ in the sense of (6.18) on any interval (τ, T), where $\tilde{y} \in C([\tau, +\infty); H)$ is the solution of problem (6.5) with control $\tilde{u}(\cdot)$. Moreover, in a standard way (see, e.g., the proof of Lemma 11 in [27]), one can prove that $\tilde{y}_n(s) \to \tilde{y}(s)$ strongly in H for all $s > \tau$.

Let $s > \tau$. Since $\widetilde{y}_n(s) \to \widetilde{y}(s)$ weakly in H, we have that

$$|\widetilde{y}(s)| \le \liminf |\widetilde{y}_n(s)|.$$

If we can show that $\limsup |\widetilde{y}_n(s)| \le |\widetilde{y}(s)|$, then $\lim |\widetilde{y}_n(s)| = |\widetilde{y}(s)|$ and the proof will be finished. For $t \in [\tau, T]$ with $T > s$, we set

$$J(t) = \frac{1}{2}|\widetilde{y}(t)|^2 - \int_\tau^t (f, \widetilde{y}(r))\, dr,$$

$$J_n(t) = \frac{1}{2}|\widetilde{y}_n(t)|^2 - \int_\tau^t (f, \widetilde{y}_n(r))\, dr.$$

We note that $\widetilde{y}_n(t) \to \widetilde{y}(t)$ for a.a. $t \in (\tau, T)$ and then $J_n(t) \to J(t)$ for a.a. t. First, we state that $\limsup J_n(s) \le J(s)$. Indeed, let $0 < t_k < s$ be such that $J_n(t_k) \to J(t_k)$. In view of (6.13) $J_n(t)$, is nonincreasing, so that

$$J_n(s) - J(s) \le |J_n(t_k) - J(t_k)| + |J(t_k) - J(s)|.$$

Since $J(t)$ is continuous at s, for any $\varepsilon > 0$, there exist t_k and $m_0(t_k)$ such that $J_n(t_k) - J(s) \le \varepsilon$, for all $m \ge m_0$, and the result follows. Therefore, since $\int_\tau^s (f, \widetilde{y}_n(r))\, dr \to \int_\tau^s (f, \widetilde{y}(r))\, dr$, we have $\limsup |\widetilde{y}_n(s)| \le |\widetilde{y}(s)|$.

Therefore, $(U1)$ holds.

Assume now additionally that $\eta_n \to \eta$ strongly in H. Let us prove that $\{\tilde{y}(\cdot), \tilde{u}(\cdot)\}$ is an optimal pair.

Fix an arbitrary $u(\cdot) \in \mathbf{U}_\tau$. Let $y_n(\cdot)$ be the solution of problem (6.5) with control $u(\cdot)$ and initial data $y_n(\tau) = \eta_n$. Then, of course, $y_n(\cdot) \to y(\cdot)$ in the sense of (6.18), where $y(\cdot)$ is the solution of problem (6.5) with control $u(\cdot) \in \mathbf{U}_\tau$ and initial data $y(\tau) = \eta$. Also, one can prove that $y_n \to y$ strongly in $L_2(\tau, T; V)$ for all $\tau < T$. Indeed, in a standard way, we obtain

$$\frac{1}{2}\frac{d}{dt}|y_n - y|^2 + \nu \|y_n - y\|^2 + B(u, y_n - y, y_n - y) = 0.$$

As $B(u, y_n - y, y_n - y) = 0$, we have

$$|y_n(T) - y(T)|^2 + 2\nu \int_\tau^T \|y_n - y\|^2 \, ds = |\eta_n - \eta|^2 \to 0, \text{ as } n \to \infty. \quad (6.22)$$

Further, since $\{\tilde{y}_n(\cdot), \tilde{u}_n(\cdot)\}$ is an optimal process, we have

$$J_\tau(\tilde{u}_n, \tilde{y}_n) \le J_\tau(y_n, u),$$

so that, for all $T > \tau$,

$$\int_\tau^T \|\tilde{y}_n - \tilde{u}_n\|^2 e^{-\delta p} dp \le \int_\tau^T \|y_n - u\|^2 e^{-\delta p} dp + \\ + \int_T^{+\infty} \|y_n - u\|^2 e^{-\delta p} dp - \int_T^{+\infty} \|\tilde{y}_n - \tilde{u}_n\|^2 e^{-\delta p} dp. \quad (6.23)$$

As we showed before

$$J_\tau(\tilde{u}_n, \tilde{y}_n) \le \tilde{C},$$

where $\tilde{C}$ does not depend on n. Moreover, since $\{\tilde{y}_n(\cdot), \tilde{u}_n(\cdot)\}$ is an optimal pair on $[T, +\infty)$ (see Lemma 6.2), if for any $T > \tau$ we consider the problem (6.5)–(6.7) with initial data $(T, \tilde{y}_n(T))$, the control $u(\cdot) \equiv 0 \in \mathbf{U}_T$, and the corresponding solution of (6.5) $z_n(\cdot)$ with $z_n(T) = \tilde{y}_n(T)$, then

$$\int_T^{+\infty} \|\tilde{y}_n - \tilde{u}_n\|^2 e^{-\delta p} dp \le \int_T^{+\infty} \|z_n\|^2 e^{-\delta p} dp.$$

But $z_n(\cdot)$ satisfies (6.12)–(6.14), so that for any $\varepsilon > 0$, there exists $T_1(\varepsilon)$ such that for any $T \ge T_1(\varepsilon)$, $n \ge 1$, we have

$$\int_T^{+\infty} \|\tilde{y}_n - \tilde{u}_n\|^2 e^{-\delta p} dp < \frac{\varepsilon}{2}.$$

The functions $\{y_n(\cdot)\}$ from (6.23) also satisfy (6.12)–(6.14), and $u(\cdot) \in \mathbf{U}_\tau$. Hence

$$\int_T^{+\infty} \|y_n - u\|^2 e^{-\delta p} dp \le 2 \int_T^{+\infty} \|y_n\|^2 e^{-\delta p} dp + 2 \int_T^{+\infty} \|u\|^2 e^{-\delta p} dp,$$

and for any $\varepsilon > 0$, there exists $T_2(\varepsilon, u)$ such that for all $T \geq T_2(\varepsilon, u)$, $n \geq 1$,

$$\int_T^{+\infty} \|y_n - u\|^2 e^{-\delta p} dp \; < \; \frac{\varepsilon}{2}.$$

Hence, for any $\varepsilon > 0$, there exists $T(\varepsilon, u)$ such that for all $T \geq T(\varepsilon, u)$, $n \geq 1$,

$$\int_\tau^T \|\tilde{y}_n - \tilde{u}_n\|^2 e^{-\delta p} dp \leq \int_\tau^T \|y_n - u\|^2 e^{-\delta p} dp \; + \; \varepsilon. \tag{6.24}$$

Then by (6.22), passing to the limit in (6.24), we obtain

$$\begin{aligned}\int_\tau^T \|\tilde{y} - \tilde{u}\|^2 e^{-\delta p} dp &\leq \liminf_{n\to\infty} \int_\tau^T \|\tilde{y}_n - \tilde{u}_n\|^2 e^{-\delta p} dp \\ &\leq \int_\tau^T \|y - u\|^2 e^{-\delta p} dp \; + \; \varepsilon, \;\; \forall T \geq T(\varepsilon, u).\end{aligned}$$

Thus, letting $T \to +\infty$, we obtain

$$J_\tau(\tilde{u}, \tilde{y}) \leq J_\tau(u, y) \; + \; \varepsilon, \; \forall u \in \mathbf{U}_\tau, \; \forall \varepsilon > 0,$$

and $\{\tilde{y}, \tilde{u}\}$ is an optimal pair.

The property $\Theta(t) \subset B_{R_0}$ follows from Theorem 6.1. Also, by Lemma 6.2 and Theorem 6.1, we obtain that $\Theta(t) = U(t, s, \Theta(s))$, for all $t \geq s$. □

Further, we shall obtain that a uniform global attractor exists and that it contains the pullback attractor $\Theta(t)$.

However, we introduce here a slightly different concept of uniform attractor for the process U.

Definition 6.3. The compact set $\mathscr{A}$ is called a uniform global attractor for the multivalued process U if

$$\lim_{t\to+\infty} \sup_{\tau\in\mathbf{R}} dist(U(t + \tau, \tau, B), \mathscr{A})) = 0, \; \forall \; B \in \beta(H), \tag{6.25}$$

and it is the minimal closed set satisfying this property.

Let us define now the multivalued map $G : \mathbf{R}^+ \times H \to P(H)$ given by

$$G(t, y_0) = \cup_{\tau\in\mathbf{R}} U(t + \tau, \tau, y_0).$$

Lemma 6.3. *G is a multivalued semiflow in the sense of Definition 1.1.*

Proof. It is clear that $G(0,\cdot) = Id$. Let $\xi \in G(t+s, y_0)$. Then for some $\tau \in \mathbf{R}$,

$$\xi \in U(t+s+\tau, \tau, y_0) \subset U(t+s+\tau, s+\tau, U(s+\tau, \tau, y_0))$$
$$\subset G(t, G(s, y_0)).$$

Hence, $G(t+s, y_0) \subset G(t, G(s, y_0))$. □

Theorem 6.3. *For the multivalued process U, there exists a uniform global attractor $\mathscr{A}$. Moreover, $\Theta(t) \subset \mathscr{A}$, for all $t \in \mathbf{R}$.*

Proof. In view of (6.21), we have that for all $B \in \beta(H)$, there exists $T(B)$ such that $G(s, B) \subset B_{R_0+1}$ if $s \geq T$. Hence,

$$G(t, B) \subset G(1, G(t-1, B)) \subset G(1, B_{R_0+1}), \forall t \geq T(B). \tag{6.26}$$

We will show that the set $G(1, B_{R_0+1})$ is precompact. If $\xi_n \in G(1, B_{R_0+1})$, then there exist $\tau_n \in \mathbf{R}$, $\eta_n \in B_{R_0+1}$ and optimal pair $\{\tilde{y}_n(\cdot), \tilde{u}_n(\cdot)\}$ of problem (6.5)–(6.7) with $\tilde{u}_n(\cdot) \in \mathbf{U}_{\tau_n}$ such that $\xi_n = \tilde{y}_n(\tau_n + 1)$, $\tilde{y}_n(\tau_n) = \eta_n$. We have that $\{\tilde{y}_n(\cdot)\}$ satisfy (6.12)–(6.14), so

$$\sup_{s\in[\tau_n, \tau_n+1]} |\tilde{y}_n(s)| \leq C_1,$$
$$\int_{\tau_n}^{\tau_n+1} \|\tilde{y}_n(s)\|^2 \, ds \leq C_2, \text{ for all } n.$$

Also, by $\|B(\tilde{u}_n, \tilde{y}_n)\|_{V^*} \leq c_1 \|\tilde{y}_n\| \|\tilde{u}_n\|_{\mathbf{L}_4} \leq c_1 \alpha \|\tilde{y}_n\|$, we have

$$\int_{\tau_n}^{\tau_n+1} \left\| \frac{d\tilde{y}_n}{dt} \right\|_{V^*}^2 ds \leq C_3.$$

Arguing as in Theorem 6.2, we obtain also that

$$\int_{\tau_n}^{\tau_n+1} \|\tilde{u}_n(s)\|^2 \, ds \leq C_4.$$

Indeed,

$$J_{\tau_n}(\tilde{u}_n, \tilde{y}_n) \leq J_{\tau_n}(0, z_n) \leq \sum_{k=0}^{\infty} \int_{\tau_n+k}^{\tau_n+k+1} \|z_n(p)\|^2 e^{-\delta p} dp \leq e^{-\delta \tau_n} \tilde{C},$$

where the constant $\tilde{C}$ does not depend on n and $z_n(\cdot)$ is the solution of (6.5) corresponding to $u \equiv 0 \in U_{\tau_n}$ and $z_n(\tau_n) = \eta_n$. So

$$e^{-\delta(\tau_n+1)}\int_{\tau_n}^{\tau_n+1}\|\tilde{u}_n(p)\|^2dp \le \int_{\tau_n}^{\tau_n+1}\|\tilde{u}_n(p)\|^2e^{-\delta p}dp \le 2\int_{\tau_n}^{\tau_n+1}\|\tilde{y}_n(p)\|^2e^{-\delta p}dp$$

$$+2\int_{\tau_n}^{\tau_n+1}\|\tilde{y}_n-\tilde{u}_n\|^2e^{-\delta p}dp \le 2e^{-\delta\tau_n}(C_2+\tilde{C}),$$

and we obtain the required estimate.

Let us define the functions $y_n(t)=\tilde{y}_n(t+\tau_n)$, $u_n(t)=\tilde{u}_n(t+\tau_n)$, $t\ge 0$. The functions $y_n(\cdot)$ are the unique solutions of problem (6.5) with initial data $y_n(0)=\eta_n$ and control $u_n(\cdot)$. By the previous estimates, there exist functions $y(\cdot)$ and $u(\cdot)$ for which the convergences (6.18) hold in the interval $(0,1)$. Also, $u\in \mathbf{U}_0$ and

$$\frac{dy_n}{dt}\to\frac{dy}{dt}\text{ weakly in }L_2(0,1;V^*).$$

Hence, by standard arguments, we obtain that $y\in C([0,1],H)$ and that it is the solution of (6.5) with control $u(\cdot)$ and initial data $y(0)=\eta$. Then in a standard way (see the proof of Theorem 6.2), one can prove that $y_n(s)\to y(s)$ strongly in H for all $s\in(0,1]$. Hence, $\xi_n=\tilde{y}_n(\tau_n+1)=y_n(1)$ is convergent and $G(1,B_{R_0+1})$ is pre-compact.

By Theorem 1.1, the omega-limit set of any $B\in\beta(H)$ given by $\omega(B)=\cap_{s\ge 0}\overline{\cup_{t\ge s}G(t,B)}$ is nonempty, compact, and attracts B, that is,

$$dist(G(t,B),\omega(B))\to 0\text{ as }t\to\infty.$$

Since $G(p+s,B)\subset G(p,B_{R_0+1})$, for all $p\ge 0$, $s\ge T(R_0)$, we deduce that $\omega(B)\subset\omega(B_{R_0+1})$ for all $B\in\beta(H)$. Then the set

$$\mathscr{A}=\cup_{B\in\beta(H)}\omega(B)=\omega(B_{R_0+1})$$

is compact and attracts every $B\in\beta(H)$. Hence,

$$\lim_{t\to+\infty}\sup_{\tau\in\mathbf{R}} dist(U(t+\tau,\tau,B),\mathscr{A}))=0. \tag{6.27}$$

The set $\mathscr{A}$ is the minimal closed set satisfying (6.27). Indeed, let $\mathscr{C}$ be a closed set satisfying (6.27) with $B=B_{R_0+1}$ and such that $\mathscr{A}\not\subset\mathscr{C}$. Then there exists $y\in\mathscr{A}$ such that $y\notin\mathscr{C}$. We take sequences y_n, t_n such that $y_n\in G(t_n,B_{R_0+1})$, so that $y_n\in U(t_n+\tau_n,\tau_n,B_{R_0+1})$ for some $\tau_n\in\mathbf{R}$, and converging to y as $n\to\infty$. Since

$$dist(y_n,\mathscr{C})\to 0\text{ as }n\to\infty,$$

we have $y\in\mathscr{C}$, which is a contradiction.

Thus, $\mathscr{A}$ is a uniform global attractor.
Finally, since $\Theta(\tau) \subset B_{R_0}$, for all $\tau \in \mathbf{R}$, we obtain that

$$\begin{aligned}\Theta(t) \subset U(t, t-s, \Theta(t-s)) \subset U(t, t-s, B_{R_0}) = U(\tau_s + s, \tau_s, B_{R_0}) \\ \subset G(s, B_{R_0}) \to \mathscr{A} \text{ as } s \to +\infty.\end{aligned}$$

Hence, $\Theta(t) \subset \mathscr{A}$, for all $t \in \mathbf{R}$. □

6.3 Relationship with the Attractor of the 3D Navier–Stokes System

Consider now the three-dimensional (3D) Navier–Stokes system

$$\begin{cases} \frac{dy}{dt} + Ay + B(y, y) = f, \\ y(\tau) = y_\tau \in H. \end{cases} \tag{6.28}$$

Our aim now is to study the relation between the pullback attractor of the optimal control problem (6.5)–(6.7) and the global attractor for the multivalued semiflow generated by (6.28). We recall first some conditional results proved in [27].

Assumption 1 Assume that for any $\tau \in \mathbf{R}$, $y_\tau \in V$, there exists a unique globally defined strong solution $y(\cdot)$ of the 3D Navier–Stokes system, that is,

$$y(\cdot) \in C([\tau, +\infty); V) \cap L_2^{loc}(\tau, +\infty; D(A)).$$

Then following [27], one can correctly define the map $G : \mathbf{R}^+ \times H \mapsto P(H)$ by

$$\begin{aligned} G(t, y_0) = \big\{ y(t) \ : \ y(\cdot) \text{ is a weak solution of (6.28)} \\ \text{with } y(0) = y_0 \text{ such that } y(\cdot) \text{ satisfies (6.13)} \big\}. \end{aligned} \tag{6.29}$$

We state a result about existence of regular solutions.

Theorem 6.4. *[27, Theorem 5] Let Assumption 1 hold. Then for any $R > 0$ and $y_0 \in H$ such that $|y_0| < R$, there exists at least one weak solution of (6.28) such that*

$$y(\cdot) \in C([0, +\infty), H), \tag{6.30}$$

$$y(\cdot) \in L_\infty([s, T]; \mathbf{L}_4(\Omega)), \text{ for all } 0 < s < T, \tag{6.31}$$

$$\|y(t)\|_{\mathbf{L}_4(\Omega)} \leq G(R, T, \delta), \tag{6.32}$$

for all $T > 0$, $0 < \delta < T$ *and for a.a.* $t \in (\delta, T)$, *where* $R \mapsto G(R, T, \delta)$, $T \mapsto G(R, T, \delta)$ *are nondecreasing functions. Moreover, (6.13) holds.*

Recall that a bounded complete trajectory of (6.28) is a weak solution defined on $(-\infty, \infty)$, which satisfies (6.13), and such that $|y(t)| \leq C$, for all $t \in (-\infty, \infty)$.

Theorem 6.5. *[27, p.261–262] Under Assumption 1, the multivalued map (6.29) is a multivalued semiflow which has a strictly invariant global attractor A, consisting of all bounded complete trajectories, that is,*

$$A = \{\varphi(t) \ : \ \varphi(\cdot) \ \textit{is a bounded weak solution of (6.28)} \\ \textit{satisfying (6.13) and defined on } (-\infty, +\infty)\}, \tag{6.33}$$

where $t \in \mathbf{R}$ *can be chosen arbitrarily.*

Now, we are ready to prove that the global attractor A of (6.28) and the pullback attractor $\Theta(t)$ of (6.5)–(6.7) coincide.

Theorem 6.6. *Under Assumption 1, there exists* $C(R_0)$ *such that if* $\alpha \geq C(R_0)$ *in (6.6), then* $A = \Theta(t)$, $\forall t \in \mathbf{R}$.

Proof. First, we shall check that $A \subset \Theta(t)$, for any $t \in \mathbf{R}$. We know from [27, p.259] that the ball $B_\delta = \{y \in H : |y| \leq R_0 + \delta\}$ is absorbing for G for any $\delta > 0$, and then, it is clear that $A \subset B_{R_0}$. Also, it follows from Theorem 6.4 the existence of $C > 0$ such that $\|\xi\|_{\mathbf{L}_4} \leq C$, for any $\xi \in A$. Indeed, since $A = G(1, A)$, we can choose a weak solution $y(\cdot)$ of the Navier–Stokes system (6.28) satisfying (6.13) and such that $y(0) \in A$, $y(1) = \xi$. This solution is unique in the class of weak solutions satisfying (6.13) [27, p.262]. Then (6.32) implies that

$$\|y(t)\|_{\mathbf{L}_4(\Omega)} \leq G\left(R_0, 1, \frac{1}{2}\right), \text{ for all } \frac{1}{2} \leq t \leq 1.$$

Choosing $C = G\left(R_0, 1, \frac{1}{2}\right)$, we obtain the desired property.

Let us take $\alpha \geq C$ in (6.6). Then for any $\xi \in A$, $t \geq \tau$, there exists $\eta \in A$ and a weak solution $y(\cdot)$ of the 3D Navier–Stokes system (6.28) satisfying (6.13) such that $\xi = y(t)$, $\eta = y(\tau)$, $y(p) \in A$, for all $p \geq \tau$. So, $\|y(p)\|_{\mathbf{L}^4} \leq C$ and $y(\cdot)$ can be taken as a control, that is, $y(\cdot) \in \mathbf{U}_\tau$. Thus, $y(\cdot)$ is in fact the optimal control, because for $\tilde{u}(\cdot) \equiv y(\cdot)$, $\tilde{y}(\cdot) \equiv y(\cdot)$, we have $J_\tau(\tilde{u}, \tilde{y}) = 0$. So, $\xi = y(t) \in U(t, \tau, \eta) \subset U(t, \tau, A)$ and taking $\tau \to -\infty$, we deduce that $\xi \in \Theta(t)$.

Further, we shall prove that $\Theta(t) \subset A$, for any $t \in \mathbf{R}$. Let $\xi \in \Theta(t)$. Since $\Theta(t) = U(t, s, \Theta(s))$, for all $t > s$, there exists an optimal pair $\{\tilde{y}(\cdot), \tilde{u}(\cdot)\}$ of problem (6.5)–(6.7) with $\tilde{y}(s) = y_s \in \Theta(s)$, $\tilde{u}(\cdot) \in \mathbf{U}_s$, such that $\tilde{y}(t) = \xi$.

We take $s < t - 2$. In view of (6.14) and $|y_s| \leq R_0$, there exists $s < s_0 < t - 1$ such that

$$\|\tilde{y}(s_0)\|^2 \leq \frac{R_0^2}{\nu} + \frac{|f|^2}{\nu^2 \lambda_1} = R_1^2. \tag{6.34}$$

It is clear that $\tilde{y}(s_0) \in V \cap \Theta(s_0)$. From the arguments in Lemma 6.2 $\tilde{u}(\cdot) \in \mathbf{U}_{s_0}$ and the pair $\{\tilde{y}(\cdot), \tilde{u}(\cdot)\}$ is also an optimal pair of (6.5)–(6.7) in $[s_0, +\infty)$. But by Assumption 1 for the initial data $\tilde{y}(s_0)$, there exists a unique globally defined strong solution $y_0(\cdot)$ of the Navier–Stokes system (6.28). We shall check that $\{y_0(\cdot), y_0(\cdot)\}$ is an optimal process of the problem (6.5)–(6.7) with $y_0(s_0) = \tilde{y}(s_0)$, $y_0(\cdot) \in \mathbf{U}_{s_0}$.

First, we note that any strong solution of (6.5) satisfies (6.12). Using this inequality and $|\tilde{y}(s_0)| \le R_0$, it is obvious that $|y_0(r)| \le R_0$ for all $r \ge s_0$.

It is known [38, p.382] that

$$\sup_{\|y_{s_0}\| \le R_1, t\in[s_0,T]} \|y(t)\| = K(R_1, T - s_0) < +\infty, \tag{6.35}$$

where $y(\cdot)$ is the unique strong solution of system (6.5) corresponding to y_{s_0}. We note that the function K is nondecreasing with respect to both variables. Then, (6.34) and (6.35) imply

$$\|y_0(r)\| \le K(R_1, 2) \text{ for all } s_0 \le r \le s_0 + 2.$$

Since $y_0(\cdot)$ also satisfies (6.14), one can choose $\widetilde{s_0} \in (s_0 + 1, s_0 + 2)$ such that $y_0(\widetilde{s_0})$ satisfies (6.34). Hence, using again (6.35), we obtain that $\|y_0(r)\| \le K(R_1, 2)$ for all $s_0 \le r \le s_0 + 3$. Repeating the same arguments inductively, we obtain that $\|y_0(r)\| \le K(R_1, 2)$ for all $r \ge s_0$. Therefore,

$$\|y_0(r)\|_{\mathbf{L}_4} \le cK(R_1, 2) \text{ for all } r \ge s_0.$$

Finally, $\int_{s_0}^{+\infty} \|y_0(r)\|^2 e^{-\delta r} dr$ follows from (6.14).

We choose $\alpha \ge \max\{G\left(R_0, 1, \frac{1}{2}\right), cK(R_1, 2)\}$. Then $y_0(\cdot) \in \mathbf{U}_{s_0}$. Since $J_{s_0}(y_0, y_0) = 0$, the pair $\{y_0(\cdot), y_0(\cdot)\}$ is an optimal process.

It follows that $J_{s_0}(\tilde{y}, \tilde{u}) = J_{s_0}(y_0, y_0) = 0$, so that $\tilde{y}(\cdot) = \tilde{u}(\cdot)$ and $\tilde{y}(\cdot)$ is a weak solution of the Navier–Stokes system (6.28). As $\tilde{y}(\cdot)$ is unique in the class of weak solutions satisfying $\tilde{y}(\cdot) \in L_8^{loc}(s_0, +\infty; \mathbf{L}_4(\Omega))$ [37, p.297–298], we have $y_0 = \tilde{y}$ on $[s_0, +\infty)$ and then $y_0(t) = \xi$. We note that $\Theta(s) = U(s, s_0, \Theta(s_0))$, for all $s \ge s_0$, implies that $y_0(s) \in \Theta(s)$, for all $t \ge s \ge s_0$. In the same way for some $s_1 < s_0 - 2$, we can define a weak solution (in fact strong) $y_1(\cdot)$ such that $y_1(s_0) = y_0(s_0)$, $y_1(s) \in \Theta(s)$, for all $s_1 \le s \le s_0$. The same can be done for some sequence $s_0 > s_1 > s_2 > \ldots > s_n \to -\infty$. Concatenating the functions $y_k(\cdot)$, we obtain a weak solution $y(\cdot)$ of (6.28) defined on $(-\infty, t]$ and such that $y(t) = \xi$, $y(s) \in \Theta(s)$, for all $s \le t$. It is easy to see that y satisfies (6.13). On the other hand, in the interval $[t, +\infty)$, we take an optimal process $\{\tilde{y}(\cdot), \tilde{u}(\cdot)\}$ of (6.5)–(6.7) such that $\tilde{y}(t) = \xi$ and in the same way one can check that $\tilde{y}(\cdot)$ is the unique strong solution of (6.28) with $\tilde{y}(t) = \xi$. Hence, we put $y(s) = \tilde{y}(s)$ for $s \ge t$. The invariance property $\Theta(s) = U(s, t, \Theta(t))$ implies that $y(s) \in \Theta(s)$, for all $s \geq t$. By Theorem 6.2, we have $\Theta(s) \subset B_{R_0}$, so that the function $y(\cdot)$ is bounded on **R**. It follows from (6.33) that $\xi = y(t) \in A$.

Therefore, $A = \Theta(t)$, for all $t \in \mathbf{R}$. □

6.4 Applications for Hydrodynamic Problems

Let us consider the problem from [41, Appendix A], [21–25]. There, we considered some hydrodynamical application to differential-operator equations in infinite-dimensional spaces. We developed a coupled Larangian–Eulerian numerical scheme for modeling the laminar flow of viscous incompressible fluid past a square prism at moderate Reynolds numbers. Then we solved the two-dimensional Navier–Stokes equations with the vorticity-velocity formulation that can be reduced to differential-operator equation. The convection step simulated by motion of Lagrangian vortex elements and diffusion of vorticity will be calculated on the multilayered adaptive grid. To reduce the dynamic loads on the body, the passive control techniques using special thin plates will be proposed. The plates will be installed either on the windward side of prism or in its wake. In the first case, the installation of a pair of symmetrical plates produces substantial decreasing the intensity of the vortex sheets separating in the windward corners of prism. In the second case, the wake symmetrization is achieved with the help of a long plate abutting upon the leeward surface. Both the ways bring narrowing of the wake and, as a result, decrease of the dynamic loads. With optimal parameters of the control system, the drag reduction has be shown to decrease considerably.

Let us consider a two-dimension laminar flow of viscous incompressible fluid past a square cylinder. Assuming critical parameters are the remote flow velocity U_∞ and the length of the side of a square a, we obtain the following dimensionless Navier–Stokes equations:

$$\frac{\partial V}{\partial t} + (V \cdot \nabla)V = -\triangle p + \frac{1}{\mathrm{Re}} \nabla^2 V, \tag{6.36}$$

$$\nabla \cdot V = 0, \tag{6.37}$$

where $V(x, y, t)$ is the fluid velocity, $p(x, y, t)$ is the pressure, ν is the fluid viscosity, and $\mathrm{Re} = U_\infty a/\nu$.

Performing the operation *rotor* with respect to each term of the (6.36) and putting the vorticity $\omega = \nabla \times V$ in view of (6.37), we obtain the equation describing evolution and diffusion of vorticity in the considered domain. Particularly, for two-dimensional problems we have

$$\frac{\partial \omega}{\partial t} + (V \cdot \nabla)\omega = \frac{1}{Re} \triangle \omega. \tag{6.38}$$

Equation (6.38) implies: if in the time point t velocity and vorticity are given, then the vorticity distribution in the next time point $t + \Delta t$ can be found. Thereafter, under obtained values ω, using Biot-Savart formula and taking into account boundary conditions on the body surface, we can find new velocity values in the domain. This computation cycle, which firstly was described by Lighthill [32], is the foundation of the vortex method. The distinctive feature of numerical algorithms based on this cycle consists in the way how diffusion and vorticity convection are calculated and also in different approaches to modeling of vorticity generation on the body wall.

To solve the diffusion problem for vorticity ω, it is necessary to fulfill boundary conditions of the body and boundary conditions of infinity. For velocity $V(x, y, t) = V_n(x, y, t) + V_\tau(x, y, t)$, these are the standard non-percolation and adhesion conditions:

$$V_n(x, y, t)|_L = U_{n\,body}, \tag{6.39}$$

$$V_\tau(x, y, t)|_L = U_{\tau\,body}, \tag{6.40}$$

where $U_{body} = U_{n\,body} + U_{\tau\,body}$ is the body velocity consisting of translation and rotation velocities in general case and L is the boundary of the body.

The choice of a boundary condition on the surface of the body L for the function ω is a nontrivial problem. It is associated with the way one describes the vorticity generation on the body wall. Due to Lighthill's method, which is effectively used in the modeling of discrete vortexes, the body surface is replaced by vortex sheet. In this case, the vorticity value on the wall ω_0 depends on intensity of the vortex sheet γ. There are different approaches to finding a function ω. One of them is based on the fact that a jump of tangential velocity (V_τ) in incompressible ideal fluid crossing the vortex sheet is equal to $\gamma/2$. Then under the adhesion condition, on the surface L, the following relationship must be fulfilled:

$$V_\tau^0 + \gamma/2 = 0.$$

Taking into account that $\omega_0 = \gamma/h$, (h is a given short distance from the wall along the normal line or an appropriate sampling interval for computational grid associated with the body) we have:

$$\omega_0 = -\frac{2V_\tau^0}{h}\bigg|_L . \tag{6.41}$$

The velocity V_τ in formula (6.41) is calculated directly on the wall (Figs. 6.1 and 6.2).

In [40], the magnitude ω_0 was determined using the expansion of tangential velocity into Taylor's series near the body surface. In this case, taking for instance the horizontal wall, we have

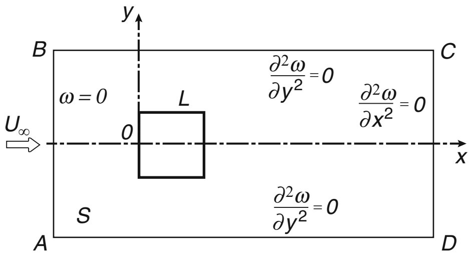

Fig. 6.1 Computation domain [41, Appendix A]

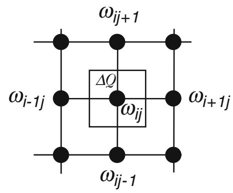

Fig. 6.2 Grid element

$$\omega_0 = -\frac{2V_\tau(x,h/2)}{h} + \left.\frac{\partial^2 V_\tau}{\partial y^2}\right|_{y=0} h/4 + O(h^2) + \ldots . \tag{6.42}$$

If in formula (6.42) only terms containing first-order infinitesimals are left, we obtain an expression which is an analogue of well-known Thompson's formula in $\omega - \psi$ model. Expressions (6.41) and (6.42) are the examples of Dirichlet condition on the function ω_0. In the work [30] for vorticity flow, Neumann condition is used. It should be noted that there is no rigorous mathematical substantiation of boundary conditions for the function ω which would correlate the intensity of the vortex sheet around the body with vorticity generated by its walls. The choice of boundary condition for the function ω essentially depends on the numerical method used for solving vortex transfer (6.38).

Let us consider an unbounded fluid flow. Then for fluid velocity perturbations caused by the body, the damping condition is satisfied:

$$V(x,y,t) \to U_\infty, \quad if \;\; r = \sqrt{x^2 + y^2} \to \infty. \tag{6.43}$$

The problem formulation is supplemented with initial conditions:

$$\begin{aligned} V(x,y,0) &= U_\infty(x,y), \\ \omega(x,y,0) &= \nabla \times U_\infty(x,y). \end{aligned} \tag{6.44}$$

In [41, Appendix A], it presented numerical algorithm, its approbation, computation of hydrodynamic loads, and algorithms for flow past a square cylinder structure control. But here, by using previous results, we can simplify conclusions. For a long-time forecast, we can investigate only the rest points of this problem. The state functions will be exponentially tends to compact invariant set in the phase space (attractor) uniformly, etc.

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Chapter 7
Properties of Resolving Operator for Nonautonomous Evolution Inclusions: Pullback Attractors

One of the most effective approaches to investigate nonlinear problems, represented by partial differential equations, inclusions and inequalities with boundary values, consists in the reduction of them into differential-operator inclusions, in infinite-dimensional spaces governed by nonlinear operators. In order to study these objects, the modern methods of nonlinear analysis have been used [7, 10, 11, 26]. Convergence of approximate solutions to an exact solution of the differential-operator equation or inclusion is frequently proved on the basis of the property of monotony or pseudomonotony of the corresponding operator. In applications, as an example of a pseudomonotone operator, the sum of a radially continuous monotone bounded operator and a strongly continuous operator was considered in [11]. Concrete examples of pseudomonotone operators were obtained by taking elliptic differential operators for which only the terms containing the highest derivatives satisfy the monotony property [26]. The papers [4, 5] became classical in the given direction of investigations. In particular, in the work [5], the class of generalized pseudomonotone operators was introduced. Let W be real Banach space continuously embedded in the real reflexive Banach space Y with dual space Y^*, and let $\langle \cdot, \cdot \rangle_Y : Y^* \times Y \to \mathbf{R}$ be the pairing between Y and Y^*. Further, by $C_v(Y^*)$, we denote the family of all nonempty closed convex bounded subsets of the space Y^*. The multivalued map $A : Y \to C_v(Y^*)$ is said to be *generalized pseudomonotone on* W if for each pair of sequences $\{y_n\}_{n\geq 1} \subset W$ and $\{d_n\}_{n\geq 1} \subset Y^*$ such that $d_n \in A(y_n)$, $y_n \to y$ weakly in W, $d_n \to d$ weakly in Y^*, from the inequality

$$\limsup_{n\to\infty} \langle d_n, y_n \rangle_Y \leq \langle d, y \rangle_Y$$

it follows that $d \in A(y)$ and $\langle d_n, y_n \rangle_Y \to \langle d, y \rangle_Y$. I.V. Skrypnik's idea of passing to subsequences in classical definitions [39], which was used for stationary and evolution inclusions by several authors (see [17–20, 23, 24, 28, 29, 46, 47] and the citations there), enabled to consider the class of w_{λ_0}-pseudomonotone maps, which includes, in particular, the class of generalized pseudomonotone on W multivalued

M.Z. Zgurovsky et al., *Evolution Inclusions and Variation Inequalities for Earth Data Processing III*, Advances in Mechanics and Mathematics 27,
DOI 10.1007/978-3-642-28512-7_7,

operators. Also, the sum of two w_{λ_0}-pseudomonotone maps continues to be w_{λ_0}-pseudomonotone. Let us remark that any multivalued map $A : Y \to C_v(Y^*)$ naturally generates the *upper* and, accordingly, *lower form*:

$$[\,A(y), \omega]_+ = \sup_{d \in A(y)} \langle d, w \rangle_Y,$$

$$[\,A(y), \omega]_- = \inf_{d \in A(y)} \langle d, w \rangle_Y, \quad y, \omega \in X.$$

The properties of the given functionals have been investigated by M.Z. Zgurovsky and V.S. Mel'nik (see [24, 28, 46]). Thus, together with the classical coercivity condition for single-valued maps, that is,

$$\frac{\langle A(y), y \rangle_Y}{\|y\|_Y} \to +\infty, \qquad \text{as} \qquad \|y\|_Y \to +\infty,$$

which ensures some important a priori estimates, one can define the property of $+$-coercivity (and, accordingly, $-$-coercivity) for multivalued maps, that is,

$$\frac{[A(y), y]_{+(-)}}{\|y\|_Y} \to +\infty, \qquad \text{as} \qquad \|y\|_Y \to +\infty.$$

Clearly, $+$-coercivity is a weaker condition than $-$-coercivity.

The recent development of the monotony method in the theory of differential-operator inclusions and evolutionary variational inequalities ensures existence of solutions of the associated equations if the operator is $-$-coercivity, quasibounded, and generalized pseudomonotone (see, e.g., [6, 9, 12–14, 32] and the citations there). Further, the results of V.S. Mel'nik [30] allowed to consider evolution inclusions with $+$-coercive w_{λ_0}-pseudomonotone quasibounded multimappings (see [17–20, 23, 24, 44, 48, 49] and the citations there).

In this chapter, we introduce a differential-operator scheme for the investigation of noncoercive nonlinear boundary-value problems for which the terms of the operator corresponding to the highest derivatives do not satisfy a monotony condition. In this framework, the properly S_k of a multivalued operator will be essential. In this way, we obtain a new theorem of existence of solutions for evolution inclusions.

A multivalued map $A : Y \to C_v(X^*)$ satisfies the *property* S_k *on* W, if for any sequence $\{y_n\}_{n \geq 0} \subset W$ such that $y_n \to y_0$ weakly in W, $d_n \to d_0$ weakly in Y^* as $n \to +\infty$, where $d_n \in A(y_n)$, $\forall n \geq 1$, from

$$\lim_{n \to \infty} \langle d_n, y_n - y_0 \rangle_Y = 0,$$

it follows that $d_0 \in A(y_0)$.

Now, we consider the simple example of S_k type operator. Let $\Omega = (0, 1)$, $Y = H_0^1(\Omega)$ be the real Sobolev space with dual space $Y^* = H^{-1}(\Omega)$ (see for details [11]). Let $A : Y \times [-1, 1] \to Y^*$ be defined by the rule

$$A(y,\alpha) = -\frac{d}{dx}\left(\alpha \frac{d}{dx} y\right).$$

Then the multivalued map

$$\mathscr{A}(y) = \{A(y,\alpha) \,|\, \alpha \in [-1,1]\}, \; y \in Y,$$

satisfies the property S_k; it is $+$-coercive but not $-$-coercive; it is not generalized pseudomonotone and $(-\mathscr{A})$ is not generalized pseudomonotone too (see [15] for details). We remark that stationary inclusions for multimappings with the S_k property were considered by V.O. Kapustyan, P.O. Kasyanov, and O.P. Kogut [15], whereas evolution inclusions for $+$-coercive w_{λ_0}-pseudomonotone quasibounded maps were studied in [17–20, 23, 24] and [44, 48, 49].

We note that the results of this chapter, are new for evolution equations too.

7.1 New Theorems for Existence Solutions for Skrypnik's Type Operators

In this section, we introduce the differential-operator scheme for investigation nonlinear boundary-value problems with summands complying with highest derivatives are not satisfied monotone condition. A multivalued map $A : Y \to C_v(X^*)$ satisfies the *property* S_k *on* W, if for any sequence $\{y_n\}_{n\geq 0} \subset W$ such that $y_n \to y_0$ weakly in W, $d_n \to d_0$ weakly in Y^* as $n \to +\infty$, where $d_n \in A(y_n)$ $\forall n \geq 1$, from

$$\lim_{n\to\infty} \langle d_n, y_n - y_0 \rangle_Y = 0,$$

it follows that $d_0 \in A(y_0)$. Now, we consider the simple example of S_k type operator. Let $\Omega = (0,1)$, $Y = H_0^1(\Omega)$ be the real Sobolev space with dual space $Y^* = H^{-1}(\Omega)$ (see for details [11]). Let $A : Y \times [-1,1] \to Y^*$ defined by the rule

$$A(y,\alpha) = -\frac{d}{dx}\left(\alpha \frac{d}{dx} y\right).$$

Then the multivalued map

$$\mathscr{A}(y) = \{A(y,\alpha) \,|\alpha \in [-1,1]\}, \; y \in Y$$

satisfies the property S_k; it is $+$-coercive; but it is not $-$-coercive, it is not generalized pseudomonotone and $(-\mathscr{A})$ is not generalized pseudomonotone too (see [15] for details). We remark that stationary inclusions for multimaps with S_k properties were considered by V.O. Kapustyan, P.O. Kasyanov and O.P. Kogut [15] and the evolution inclusions for $+$-coercive w_{λ_0}-pseudomonotone quasibounded

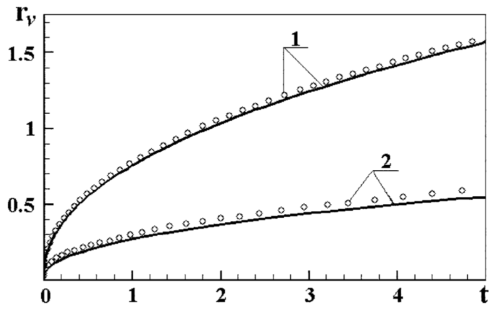

Fig. 7.1 Vortex diffusion in unbounded domain. Comparison of calculation results ○ ○ ○ for vector r_v with the exact solution ——: **1** – Re $= 10^2$, **2** – Re $= 10^3$ (see Sect. 6.4 and [41, Appendix A])

maps by V.S. Mel'nik, P.O. Kasyanov and J. Valero (see [17–20, 23, 24, 44, 48, 49] and citations there). The obtained in this paper results are new results for evolution equations too (Fig. 7.1).

In works [42, 43] and papers [16–20, 22, 25–27, 38–41] differential-operator inclusions with $+$-coercive W_{λ_0}-pseudomonotone maps closed with respect to the sum are considered. In the paper [36] it is showed that in general cases the classes of λ_0-pseudomonotone maps coincide with the classes of the classical pseudomonotone maps. Therefore, results from [19, 42, 43] (references from [36]) concerned inclusions with sums of λ_0-pseudomonotone maps can be strengthened to the pseudomonotony in the classical sense. It is worth noting that works [42, 43] and papers [16–20, 22, 25–27, 38–41] contain results concerned inclusions with $+$-coercive maps, about what the authors of [36] keep silent, at that they make wrong conclusions about—coercivity of $+$-coercive map. The results of works [42, 43] and papers [16–20, 22, 25–27, 38–41] remain fundamentally new in such directions: (1) sufficiently weakened $+$-coercivity in comparison with the classical $-$-coercivity (2) possibility of considering the sum of classical pseudomonotone maps (3) justification of new methods of search of approximate solution for such objects.

In fact, the conditions of theorems that the authors of work [36] refer to in the majority of cases (as it is showed in this book) allow us to study the dynamics of all weak solutions as in $t \to \infty$ of evolution equations and inclusions, the structure of the global and trajectory attractors for m-semiflows for all weak solutions. In this book it is showed that, for example, in the autonomous case all weak solutions are uniformly attracted to the compact in the phase space invariant global attractor. The theory of differential-operator inclusions is not exhausted by these cases. Therefore, we introduce, in particular, in this chapter, the results on inclusions with conditions that contain as partial cases the classical pseudomonotone, coercivity etc. Results of this section are borrowed from [16, 21, 22, 43, 45].

7.1.1 Problem Definition

Let $(V_1, \|\cdot\|_{V_1})$ and $(V_2, \|\cdot\|_{V_2})$ be some reflexive separable Banach spaces, continuously embedded in the Hilbert space $(H, (\cdot,\cdot))$ such that

$$V := V_1 \bigcap V_2 \text{ is dense in spaces } V_1,\ V_2 \text{ and } H \tag{7.1}$$

After the identification $H \equiv H^*$, we get

$$V_1 \subset H \subset V_1^*, \qquad V_2 \subset H \subset V_2^*, \tag{7.2}$$

with continuous and dense embeddings [11], where $(V_i^*, \|\cdot\|_{V_i^*})$ is the topologically conjugate of V_i space with respect to the canonical bilinear form

$$\langle\cdot,\cdot\rangle_{V_i} : V_i^* \times V_i \to \mathbf{R}\ (i = 1, 2)$$

which coincides on $H \times V$ with the inner product $(\cdot,\cdot)$ on H. Let us consider the functional spaces

$$X_i = L_{r_i}(S; H) \cap L_{p_i}(S; V_i),$$

where $S = [0, T]$, $0 < T < +\infty$, $1 < p_i \le r_i < +\infty$ $(i = 1, 2)$. The spaces X_i are Banach spaces with the norms $\|y\|_{X_i} = \|y\|_{L_{p_i}(S;V_i)} + \|y\|_{L_{r_i}(S;H)}$. Moreover, X_i is a reflexive space.

Let us also consider the Banach space $X = X_1 \bigcap X_2$ with the norm $\|y\|_X = \|y\|_{X_1} + \|y\|_{X_2}$. Since the spaces $L_{q_i}(S; V_i^*) + L_{r_{i'}}(S; H)$ and X_i^* are isometrically isomorphic, we identify them. Analogously,

$$X^* = X_1^* + X_2^* = L_{q_1}(S; V_1^*) + L_{q_2}(S; V_2^*) + L_{r_{1'}}(S; H) + L_{r_{2'}}(S; H),$$

where $r_i^{-1} + r_{i'}^{-1} = p_i^{-1} + q_i^{-1} = 1$.

Let us define the duality form on $X^* \times X$

$$\langle f, y\rangle = \int_S (f_{11}(\tau), y(\tau))_H\, d\tau + \int_S (f_{12}(\tau), y(\tau))_H\, d\tau + \int_S \langle f_{21}(\tau), y(\tau)\rangle_{V_1} d\tau$$

$$+ \int_S \langle f_{22}(\tau), y(\tau)\rangle_{V_2} d\tau = \int_S (f(\tau), y(\tau)) d\tau,$$

where $f = f_{11} + f_{12} + f_{21} + f_{22}$, $f_{1i} \in L_{r_{i'}}(S; H)$, $f_{2i} \in L_{q_i}(S; V_i^*)$.

Following by [26], we may assume that there is a separable Hilbert space V_σ such that $V_\sigma \subset V_1$, $V_\sigma \subset V_2$ with continuous and dense embedding and $V_\sigma \subset H$ with compact and dense embedding. Then

$$V_\sigma \subset V_1 \subset H \subset V_1^* \subset V_\sigma^*, \quad V_\sigma \subset V_2 \subset H \subset V_2^* \subset V_\sigma^*$$

with continuous and dense embedding. For $i = 1, 2$, let us set

$$X_{i,\sigma} = L_{r_i}(S; H) \cap L_{p_i}(S; V_\sigma), \quad X_\sigma = X_{1,\sigma} \cap X_{2,\sigma},$$
$$X^*_{i,\sigma} = L_{r_i'}(S; H) + L_{q_i}(S; V^*_\sigma), \quad X^*_\sigma = X^*_{1,\sigma} + X^*_{2,\sigma},$$
$$W_{i,\sigma} = \{y \in X_i \mid y' \in X^*_{i,\sigma}\}, \qquad W_\sigma = W_{1,\sigma} \cap W_{2,\sigma}.$$

For multivalued (in general) map $A : X \overset{\to}{\to} X^*$, let us consider such problem:

$$\begin{cases} u' + A(u) \ni f, \\ u(0) = a, \ u \in W \subset C(S; H), \end{cases} \tag{7.3}$$

where $a \in H$ and $f \in X^*$ are arbitrary fixed elements. The goal of this work is to prove the solvability for the given problem by the Faedo–Galerkin method.

7.1.2 The Class $\mathscr{H}(X^*)$

Let us note that $B \in \mathscr{H}(X^*)$, if for an arbitrary measurable set $E \subset S$ and for arbitrary $u, v \in B$ the inclusion $u + (v - u)\chi_E \in B$ is true. Here and further for $d \in X^*$

$$(d\chi_E)(\tau) = d(\tau)\chi_E(\tau) \text{ for a.e. } \tau \in S, \quad \chi_E(\tau) = \begin{cases} 1, \ \tau \in E, \\ 0, \ \text{ else.} \end{cases}$$

Lemma 7.1. *[42] $B \in \mathscr{H}(X^*)$ if and only if $\forall n \geq 1$, $\forall \{d_i\}_{i=1}^n \subset B$, and for arbitrary measurable pairwise disjoint subsets $\{E_j\}_{j=1}^n$ of the set S: $\bigcup_{j=1}^n E_j = S$, the following $\sum_{j=1}^n d_j \chi_{E_j} \in B$ is true.*

Let us remark, that $\emptyset$, $X^* \in \mathscr{H}(X^*)$; $\forall f \in X^*$ $\{f\} \in \mathscr{H}(X^*)$; if $K : S \overset{\to}{\to} V^*$ is an arbitrary multivalued map, then

$$\{f \in X^* \mid f(t) \in K(t) \text{ for a.e. } t \in S\} \in \mathscr{H}(X^*).$$

At the same time, for an arbitrary $v \in V^* \backslash \bar{0}$ that is not equal to 0, the closed convex set $B = \{f \in X^* \mid f \equiv \alpha v, \ \alpha \in [0, 1]\} \notin \mathscr{H}(X^*)$, as $g(\cdot) = v \cdot \chi_{[0;T/2]}(\cdot) \notin B$.

7.1.3 Classes of MultiValued Maps

Let us consider now the main classes of multivalued maps. Let Y be some reflexive Banach space, Y^* be its topologically adjoint, $\langle \cdot, \cdot \rangle_Y : Y^* \times Y \to \mathbf{R}$ be the pairing and $A : Y \overset{\to}{\to} Y^*$ be the strict multivalued map, that is, $A(y) \neq \emptyset \ \forall y \in X$.

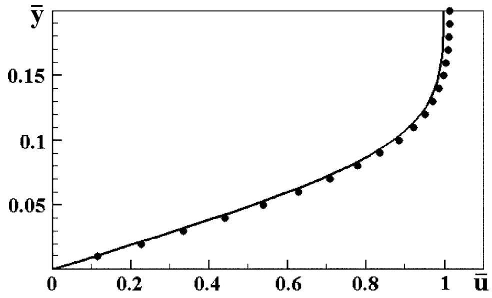

Fig. 7.2 Longitudinal velocity profile in the attached layer for the middle of the plate at $\mathrm{Re}_L = 10^3$: —— Blazius solution, *black circle* computations (see Sect. 6.4 and [41, Appendix A])

For this map, let us define the upper $\|A(y)\|_+ = \sup_{d\in\mathscr{A}(y)} \|d\|_{X^*}$ and the lower $\|A(y)\|_= \inf_{d\in\mathscr{A}(y)} \|d\|_{X^*}$ norms, where $y \in Y$. Let us consider the next maps which are connected with A : $\mathrm{co}\, A : Y \overset{\rightarrow}{\rightarrow} Y^*$ and $\overline{\mathrm{co}}\, A : Y \overset{\rightarrow}{\rightarrow} Y^*$, which are defined by the next relations $(\mathrm{co}\, A)(y) = \mathrm{co}\,(A(y))$ and $(\overline{\mathrm{co}}\, A(y)) = \overline{\mathrm{co}\,(A(y))}$, respectively, where $\overline{\mathrm{co}\,(A(y))}$ is the weak closeness of the convex hull of the set $A(y)$ in the space Y^*. It is known that strict multi-valued maps $A, B : Y \overset{\rightarrow}{\rightarrow} Y^*$ have such properties [24, 28, 47]:

1. $[A(y), v_1 + v_2]_+ \le [A(y), v_1]_+ + [A(y), v_2]_+$.
   $[A(y), v_1 + v_2]_- \ge [A(y), v_1]_- + [A(y), v_2]_- \ \forall y, v_1, v_2 \in Y$.
2. $[A(y), v]_+ = -[A(y), -v]_-$.
   $[A(y) + B(y), v]_{+(-)} = [A(y), v]_{+(-)} + [B(y), v]_{+(-)} \ \forall y, v \in Y$.
3. $[A(y), v]_{+(-)} = [\overline{\mathrm{co}} A(y), v]_{+(-)} \ \forall y, v \in Y$.
4. $[A(y), v]_{+(-)} \le \|A(y)\|_{+(-)} \|v\|_Y$, $\|A(y) + B(y)\|_+ \le \|A(y)\|_+ + \|B(y)\|_+$;

partially, the inclusions $d \in \overline{co}\, A(y)$ is true if and only if

$$[A(y), v]_+ \ge \langle d, v \rangle_Y \quad \forall v \in Y.$$

Let $D \subset Y$. If $a(\cdot, \cdot) : D \times Y \to \mathbf{R}$, then for every $y \in D$, the functional $Y \ni w \mapsto a(y, w)$ is positively homogeneous convex and lower semicontinuous if and only if there exists the multivalued map $A : Y \overset{\rightarrow}{\rightarrow} Y^*$ with the definition domain $D(A) = D$ such, that

$$a(y, w) = [A(y), w]_+ \quad \forall y \in D(A),\ \forall w \in Y.$$

Further, $y_n \rightharpoonup y$ in Y will mean, that y_n converges weakly to y in Y.

Let W be some normalized space that continuously embedded into Y. Let us consider multivalued map $A : Y \overset{\rightarrow}{\rightarrow} Y^*$ (Fig. 7.2).

Definition 7.1. The strict multivalued map $A : Y \overset{\rightarrow}{\rightarrow} Y^*$ is called:

- λ_0-*pseudomonotone on* W, if for any sequence $\{y_n\}_{n\ge 0} \subset W$ such, that $y_n \rightharpoonup y_0$ in W, $d_n \rightharpoonup d_0$ in Y^* as $n \to +\infty$, where $d_n \in \overline{\mathrm{co}}\, A(y_n)\ \forall\, n \ge 1$, from the inequality

$$\overline{\lim_{n\to\infty}} \langle d_n, y_n - y_0\rangle_Y \leq 0 \tag{7.4}$$

it follows the existence of subsequence $\{y_{n_k}, d_{n_k}\}_{k\geq 1}$ from $\{y_n, d_n\}_{n\geq 1}$, for that

$$\underline{\lim_{k\to\infty}} \langle d_{n_k}, y_{n_k} - w\rangle_Y \geq [A(y_0), y_0 - w]_- \quad \forall\, w \in Y \tag{7.5}$$

is fulfilled.

- *Bounded*, if for every $L > 0$, there exists such $l > 0$ that

$$\forall\, y \in Y : \|y\|_Y \leq L, \text{ it follows that } \|A(y)\|_+ \leq l.$$

Definition 7.2. The strict multivalued map $A : X \overset{\rightarrow}{\rightarrow} X^*$ is called:

- *The operator of the Volterra type*, if for arbitrary $u, v \in X$, $t \in S$ from the equality $u(s) = v(s)$ for a.e. $s \in [0, t]$, it follows, that $[A(u), \xi_t]_+ = [A(v), \xi_t]_+$

$$\forall \xi_t \in X : \xi_t(s) = 0 \text{ for a.e. } s \in S\backslash[0, t].$$

- $+(-)$-*coercive*, if there exists the real function $\gamma : \mathbf{R}_+ \to \mathbf{R}$ such that $\gamma(s) \to +\infty$ as $s \to +\infty$ and

$$[A(y), y]_{+(-)} \geq \gamma(\|y\|_Y)\|y\|_Y \quad \forall y \in Y$$

- *Demiclosed*, if from that fact, that $y_n \to y$ in Y, $d_n \rightharpoonup d$ in Y^*, where

$$d_n \in A(y_n),\ n \geq 1,\ \text{it follows that } d \in A(y)$$

Let us consider multivalued maps that act from X_m into X_m^*, $m \geq 1$. Let us remark that embeddings $X_m \subset Y_m \subset X_m^*$ are continuous and the embedding W_m into X_m is compact [26].

Definition 7.3. The multivalued map $\mathscr{A} : X_m \to C_v(X_m^*)$ is called (W_m, X_m^*)-weakly closed, if from that fact, that $y_n \rightharpoonup y$ in W_m, $d_n \rightharpoonup d$ in X_m^*, $d_n \in \mathscr{A}(y_n)$ $\forall n \geq 1$, it follows, that $d \in \mathscr{A}(y)$.

Lemma 7.2. *The multivalued map $\mathscr{A} : X_m \to C_v(X_m^*)$ satisfies the property S_k on W_m if and only if $\mathscr{A} : X_m \to C_v(X_m^*)$ is (W_m, X_m^*)-weakly closed.*

Proof. Let us prove the necessity. Let $y_n \rightharpoonup y$ in W_m, $d_n \rightharpoonup d$ in X_m^*, where $d_n \in \mathscr{A}(y_n)$ $\forall n \geq 1$. Then $y_n \to y$ in X_m and $\langle d_n, y_n - y\rangle_{X_m} \to 0$ as $n \to +\infty$. Therefore, in virtue of $\mathscr{A}$ satisfies the S_k property on W_m, we obtain that $d \in \mathscr{A}(y)$.

Let us prove sufficiency. Let $y_n \rightharpoonup y$ in W_m, $d_n \rightharpoonup d$ in X_m^*, $\langle d_n, y_n - y\rangle_{X_m} \leq 0$ as $n \to +\infty$, where $d_n \in \mathscr{A}(y_n)$ $\forall n \geq 1$. Then $y_n \to y$ in X_m and $d \in \mathscr{A}(y)$.

The lemma is proved. □

Corollary 7.1. *If the multivalued map $\mathscr{A} : X_m \to C_v(X_m^*)$ satisfies the property S_k on W_m, then $\mathscr{A}$ is λ_0-pseudomonotone on W_m.*

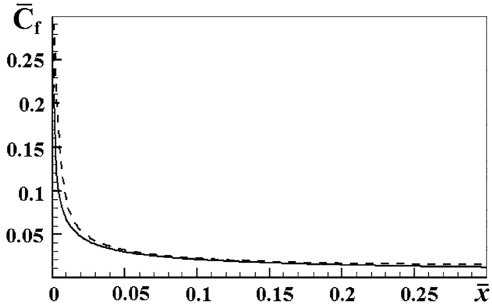

Fig. 7.3 Plate's longitude coordinate dependence of the friction coefficient at $\mathrm{Re}_L = 10^3$: —— Blazius solution, - - - computations (see Sect. 6.4 and [41, Appendix A])

7.1.4 The Main Results

In the next theorem, we will prove the solvability and justify the Faedo–Galerkin method for the problem (7.3).

Theorem 7.1. *Let $a = \bar{0}$, $A : X \to C_v(X^*) \cap \mathscr{H}(X^*)$ be $+$-coercive bounded map of the Volterra type that satisfies the property S_k on W_σ. Then for arbitrary $f \in X^*$, there exists at least one solution of the problem (7.3) that can be obtained by the Faedo–Galerkin method.*

Proof. From $+$-coercivity for $A : X \overset{\to}{\to} X^*$, it follows that $\forall\, y \in X$

$$[A(y), y]_+ \geq \gamma(\|y\|_X)\|y\|_X.$$

So, $\exists\, r_0 > 0 : \gamma(r_0) > \|f\|_{X^*} \geq 0$. Therefore,

$$\forall y \in X : \ \|y\|_X = r_0 \quad [A(y) - f, y]_+ \geq 0. \tag{7.6}$$

The solvability of approximate problems (Fig. 7.3).

Let us consider the complete vectors system $\{h_i\}_{i\geq 1} \subset V$ such that:

(α_1) $\{h_i\}_{i\geq 1}$ orthonormal in H.
(α_2) $\{h_i\}_{i\geq 1}$ orthogonal in V.
(α_3) $\forall\ i \geq 1 \ \ (h_i, v)_V = \lambda_i(h_i, v) \quad \forall v \in V$.

where $0 \leq \lambda_1 \leq \lambda_2, \ldots, \lambda_j \to \infty$ as $j \to \infty$, $(\cdot,\cdot)_V$ is the natural inner product in V, that is, $\{h_i\}_{i\geq 1}$ is a special basis [40]. Let for each $m \geq 1$ $H_m = \mathrm{span}\,\{h_i\}_{i=1}^m$, on which we consider the inner product induced from H that we again denote by $(\cdot,\cdot)$. Due to the equivalence of H^* and H, it follows that $H_m^* \equiv H_m$; $X_m = L_{p_0}(S; H_m)$, $X_m^* = L_{q_0}(S; H_m)$, $p_0 = \max\{r_1, r_2\}$, $q_0 > 1$: $1/p_0 + 1/q_0 = 1$, $\langle\cdot,\cdot\rangle_{X_m} = \langle\cdot,\cdot\rangle_X|_{X_m^*\times X_m}$, $W_m := \{y \in X_m |\ y' \in X_m^*\}$, where y' is the derivative of an element $y \in X_m$ is considered in the sense of $\mathscr{D}^*(S, H_m)$. For any $m \geq 1$, let $I_m \in \mathscr{L}(X_m; X)$ be the canonical embedding of X_m in X and I_m^* be the adjoint operator to I_m. Then

$$\forall\, m \geq 1 \quad \|I_m^*\|_{\mathscr{L}(X_\sigma^*; X_\sigma^*)} = 1. \tag{7.7}$$

Let us consider such maps [17]:

$$A_m := I_m^* \circ A \circ I_m : X_m \to C_v(X^*), \quad f_m := I_m^* f.$$

So, from (7.6) and corollary 7.1, applying analogical thoughts with [17, 20], we will obtain that:

(j_1) A_m is λ_0-pseudomonotone on W_m.
(j_2) A_m is bounded.
(j_3) $[A_m(y) - f_m, y]_+ \geq 0 \; \forall y \in X_m$: $\|y\|_X = r_0$.

Let us consider the operator $L_m : D(L_m) \subset X_m \to X_m^*$ with the definition domain

$$D(L_m) = \{y \in W_m \mid y(0) = \bar{0}\} = W_m^0$$

that acts by the rule:

$$\forall y \in W_m^0 \quad L_m y = y',$$

where the derivative y' we consider in the sense of the distributions space $\mathscr{D}^*(S; H_m)$. From [17] for the operator L_m, the next properties are true:

(j_4) L_m is linear.
(j_5) $\forall\, y \in W_m^0 \; \langle L_m y, y\rangle \geq 0$.
(j_6) L_m is maximal monotone.

Therefore, conditions (j_1)–$(j_6)$and the Theorem 3.1 from [18] guarantee the existence at least one solution $y_m \in D(L_m)$ of the problem:

$$L_m(y_m) + A_m(y_m) \ni f_m, \quad \|y_m\|_X \leq r_0$$

that can be obtained by the method of singular perturbations. This means that y_m is the solution of such problem:

$$\begin{cases} y_m' + A_m(y_m) \ni f_m \\ y_m(0) = \bar{0}, \; y_m \in W_m, \; \|y_m\|_X \leq R, \end{cases} \tag{7.8}$$

where $R = r_0$.

Passing to the limit.

From the inclusion from (7.8) it follows, that $\forall m \geq 1 \; \exists d_m \in A(y_m)$:

$$I_m^* d_m = f_m - y_m' \in A_m(y_m) = I_m^* A(y_m). \tag{7.9}$$

1. The boundedness of $\{d_m\}_{m\geq 1}$ in X^* follows from the boundedness of A and from (7.8). Therefore,

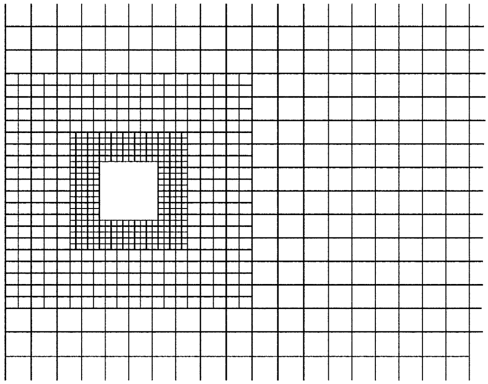

Fig. 7.4 Configuration of the grid (see Sect. 6.4 and [41, Appendix A])

$$\exists c_1 > 0 : \quad \forall m \geq 1 \quad \|d_m\|_{X^*} \leq c_1. \tag{7.10}$$

2. Let us prove the boundedness $\{y'_m\}_{m\geq 1}$ in X^*_σ. From (7.9), it follows that $\forall m \geq 1\ y'_m = I^*_m(f - d_m)$, and, taking into account (7.7), (7.8), and (7.10), we have

$$\|y'_m\|_{X^*_\sigma} \leq \|y_m\|_{W_\sigma} \leq c_2 < +\infty. \tag{7.11}$$

In virtue of (7.8) and the continuous embedding $W_m \subset C(S; H_m)$, we obtain (see [37]) that $\exists\, c_3 > 0$ such, that

$$\forall m \geq 1,\ \forall t \in S \quad \|y_m(t)\|_H \leq c_3 \tag{7.12}$$

3. In virtue of estimations from (7.10)–(7.12), due to the Banach–Alaoglu theorem, taking into account the compact embedding $W \subset Y$, it follows the existence of subsequences

$$\{y_{m_k}\}_{k\geq 1} \subset \{y_m\}_{m\geq 1}, \quad \{d_{m_k}\}_{k\geq 1} \subset \{d_m\}_{m\geq 1}$$

and elements $y \in W$, $d \in X^*$, for which the next converges take place (Fig. 7.4):

$$\begin{gathered} y_{m_k} \rightharpoonup y \text{ in } W, \quad d_{m_k} \rightharpoonup d \text{ in } X^* \\ y_{m_k}(t) \rightharpoonup y(t) \text{ in } H \text{ for each } t \in S \\ y_{m_k}(t) \to y(t) \text{ in } H \text{ for a.e. } t \in S, \text{ as } k \to \infty. \end{gathered} \tag{7.13}$$

From here, as $\forall k \geq 1\ y_{m_k}(0) = \bar{0}$, then $y(0) = \bar{0}$.

4. Let us prove that

$$y' = f - d. \tag{7.14}$$

Let $\varphi \in D(S)$, $n \in \mathbf{N}$ and $h \in H_n$. Then $\forall k \geq 1$: $m_k \geq n$, we have

$$\left(\int_S \varphi(\tau)(y'_{m_k}(\tau) + d_{m_k}(\tau))d\tau, h\right) = \langle y'_{m_k} + d_{m_k}, \psi\rangle,$$

where $\psi(\tau) = h \cdot \varphi(\tau) \in X_n \subset X$. Let us remark that here we use the property of Bochner integral [11, Theorem IV.1.8, p.153]. Since for $m_k \geq n$ $H_{m_k} \supset H_n$, then $\langle y'_{m_k} + d_{m_k}, \psi\rangle = \langle f_{m_k}, \psi\rangle$. Therefore, $\forall k \geq 1 : m_k \geq n$

$$\langle f_{m_k}, \psi\rangle = \left(\int_S \varphi(\tau) f(\tau)d\tau, h\right).$$

Hence, for all $k \geq 1$: $m_k \geq n$,

$$\left(\int_S \varphi(\tau)y'_{m_k}(\tau)d\tau,\ h\right) = \langle f - d_{m_k},\ \psi\rangle$$

$$\to \left(\int_S \varphi(\tau)(f(\tau) - d(\tau)d\tau,\ h\right) \text{ as } k \to \infty \quad (7.15)$$

The last follows from the weak convergence d_{m_k} to d in X^*.

From the convergence (7.13), we have

$$\left(\int_S \varphi(\tau)y'_{m_k}(\tau)d\tau,\ h\right) \to \big(y'(\varphi),\ h\big) \text{ as } k \to \infty \quad (7.16)$$

where

$$\forall\, \varphi \in \mathscr{D}(S)\ \ y'(\varphi) = -y(\varphi') = -\int_S y(\tau)\varphi'(\tau)d\tau.$$

Therefore, from (7.15) and (7.16), it follows that

$$\forall \varphi \in \mathscr{D}(S)\ \forall h \in \bigcup_{m\geq 1} H_m \quad (y'(\varphi), h) = \left(\int_S \varphi(\tau)(f(\tau) - d(\tau))d\tau, h\right).$$

Since $\bigcup_{m\geq 1} H_m$ is dense in V, we have that

$$\forall \varphi \in \mathscr{D}(S) \quad y'(\varphi) = \int_S \varphi(\tau)(f(\tau) - d(\tau))d\tau.$$

Therefore, $y' = f - d \in X^*$.

5. In order to prove that y is the solution of the problem (7.3), it remains to show that y satisfies the inclusion $y' + A(y) \ni f$. In virtue if identity (7.14), it is enough to prove that $d \in A(y)$.

From (7.13), it follows the existence of $\{\tau_l\}_{l\geq 1} \subset S$ such that $\tau_l \nearrow T$ as $l \to +\infty$ and

$$\forall\, l \geq 1 \quad y_{m_k}(\tau_l) \to y(\tau_l) \text{ in } H \text{ as } k \to +\infty \tag{7.17}$$

Let us show that for any $l \geq 1$,

$$\langle d, w\rangle \leq [A(y), w]_+ \quad \forall w \in X : \ w(t) = 0 \text{ for a e. } t \in [\tau_l, T]. \tag{7.18}$$

Let us fix an arbitrary $\tau \in \{\tau_l\}_{l\geq 1}$. For $i = 1, 2$, let us set

$$X_{i,\sigma}(\tau) = L_{r_i}(\tau, T; H) \cap L_{p_i}(\tau, T; V_\sigma), \quad X_\sigma(\tau) = X_{1,\sigma}(\tau) \bigcap X_{2,\sigma}(\tau),$$

$$X^*_{i,\sigma}(\tau) = L_{r_i'}(\tau, T; H) + L_{q_i}(\tau, T; V^*_\sigma), \quad X^*_\sigma(\tau) = X^*_{1,\sigma}(\tau) + X^*_{2,\sigma}(\tau),$$

$$W_{i,\sigma}(\tau) = \{y \in X_i(\tau) \mid y' \in X^*_{i,\sigma}(\tau)\}, \quad W_\sigma(\tau) = W_{1,\sigma}(\tau) \bigcap W_{2,\sigma}(\tau).$$

$$a_0 = y(\tau), \quad a_k = y_{m_k}(\tau), \quad k \geq 1.$$

Similarly, we introduce $X(\tau)$, $X^*(\tau)$, $W(\tau)$. From (7.17), it follows that

$$a_k \to a_0 \text{ in } H \text{ as } k \to +\infty. \tag{7.19}$$

For any $k \geq 1$, let $z_k \in W(\tau)$ be such that

$$\begin{cases} z_k' + J(z_k) \ni \bar{0}, \\ z_k(\tau) = a_k, \end{cases} \tag{7.20}$$

where $J : X(\tau) \to C_v(X^*(\tau))$ be the duality (in general multivalued) mapping, that is,

$$[J(u), u]_+ = [J(u), u]_- = \|u\|^2_{X(\tau)} = \|J(u)\|^2_+ = \|J(u)\|^2_-, \quad u \in X(\tau).$$

We remark that the problem (7.20) has a solution $z_k \in W(\tau)$ because J is monotone, coercive, bounded, and demiclosed (see [1, 2, 11, 18]). Let us also note that for any $k \geq 1$,

$$\|z_k(T)\|^2_H - \|a_k\|^2_H = 2\langle z_k', z_k\rangle_{X(\tau)} + 2\|z_k\|^2_{X(\tau)} = 0.$$

Hence,

$$\forall k \geq 1 \quad \|z_k'\|_{X^*(\tau)} = \|z_k\|_{X(\tau)} \leq \frac{1}{\sqrt{2}}\|a_k\|_H \leq c_3.$$

Due to (7.19), similarly to [11, 18], as $k \to +\infty$, z_k weakly converges in W to the unique solution $z_0 \in W$ of the problem (7.20) with initial time value condition $z(0) = a_0$. Moreover,

$$z_k \to z_0 \text{ in } X(\tau) \text{ as } k \to +\infty \tag{7.21}$$

because $\overline{\lim}_{k\to+\infty} \|z_k\|^2_{X(\tau)} \leq \|z_0\|^2_{X(\tau)}$, $z_k \rightharpoonup z_0$ in $X(\tau)$ and $X(\tau)$ is a Hilbert space.

For any $k \geq 1$, let us set

$$u_k(t) = \begin{cases} y_{m_k}(t), & \text{if } t \in [0,\tau], \\ z_k(t), & \text{elsewhere}, \end{cases} \qquad g_k(t) = \begin{cases} d_{m_k}(t), & \text{if } t \in [0,\tau], \\ \hat{d}_k(t), & \text{elsewhere}, \end{cases}$$

where $\hat{d}_k \in A(u_k)$ is an arbitrary. As $\{u_k\}_{k\geq 1}$ is bounded, $A : X \rightrightarrows X^*$ is bounded, then $\{\hat{d}_k\}_{k\geq 1}$ is bounded in X^*. In virtue of (7.21), (7.13), and (7.17),

$$\begin{aligned}
\lim_{k\to+\infty} \langle g_k, u_k - u\rangle &= \lim_{k\to+\infty} \int_0^\tau (d_k(t), y_k(t) - y(t))\, dt = \lim_{k\to+\infty} \int_0^\tau \big(f(t) \\
&\quad - y_k'(t), y_k(t) - y(t)\big)\, dt \\
&= \lim_{k\to+\infty} \int_0^\tau \big(y_k'(t), y(t) - y_k(t)\big)\, dt \\
&= \lim_{k\to+\infty} \frac{1}{2}\big(\|y_k(0)\|_H^2 - \|y_k(\tau)\|_H^2\big) \\
&\quad + \lim_{k\to+\infty} \int_0^\tau \big(y_k'(t), y(t)\big)\, dt \\
&= \frac{1}{2}\big(\|y(0)\|_H^2 - \|y(\tau)\|_H^2\big) + \int_0^\tau \big(y'(t), y(t)\big)\, dt = 0.
\end{aligned}$$

So,

$$\lim_{k\to+\infty} \langle g_k, u_k - u\rangle = 0. \tag{7.22}$$

Let us show that $g_k \in A(u_k)$ $\forall k \geq 1$. For any $w \in X$, let us set

$$w_\tau(t) = \begin{cases} w(t), & \text{if } t \in [0, \tau], \\ \bar{0}, & \text{elsewhere}, \end{cases} \qquad w^\tau(t) = \begin{cases} \bar{0}, & \text{if } t \in [0, \tau], \\ w(t), & \text{elsewhere}. \end{cases}$$

In virtue of A is the Volterra type operator, we obtain that

$$\begin{aligned} \langle g_k, w\rangle &= \langle d_{m_k}, w_\tau\rangle + \langle \hat{d}_k, w^\tau\rangle \\ &\leq [A(y_{m_k}), w_\tau]_+ + \langle \hat{d}_k, w^\tau\rangle \\ &= [A(u_k), w_\tau]_+ + \langle \hat{d}_k, w^\tau\rangle \\ &\leq [A(u_k), w_\tau]_+ + [A(u_k), w^\tau]_+. \end{aligned}$$

Due to $A(u_k) \in \mathscr{H}(X^*)$, similarly to [42], we obtain that

$$[A(u_k), w_\tau]_+ + [A(u_k), w^\tau]_+ = [A(u_k), w]_+.$$

As $w \in X$ is an arbitrary, then $g_k \in A(u_k)$ $\forall k \geq 1$. Due to $\{u_k\}_{k\geq 1}$ is bounded in X, then $\{g_k\}_{k\geq 1}$ is bounded in X^*. Thus, up to a subsequence $\{u_{k_j}, g_{k_j}\}_{j\geq 1} \subset \{u_k, g_k\}_{k\geq 1}$, for some $u \in W$, $g \in X^*$, the next convergence takes place

$$u_{k_j} \rightharpoonup u \text{ in } W_\sigma, \quad g_{k_j} \rightharpoonup g \text{ in } X^* \text{ as } j \to \infty \tag{7.23}$$

We remark that

$$u(t) = y(t), \quad g(t) = d(t) \text{ for a.e. } t \in [0, \tau]. \tag{7.24}$$

In virtue of (7.22), (7.23), as A satisfies the property S_k on W_σ, we obtain that $g \in A(u)$. Hence, due to (7.24), as A is the Volterra type operator, for any $w \in X$ such that $w(t) = 0$ for a.e. $t \in [\tau, T]$, we have

$$\langle d, w\rangle = \langle g, w\rangle \leq [A(u), w]_+ = [A(y), w]_+.$$

As $\tau \in \{\tau_l\}_{l\geq 1}$ is an arbitrary, we obtain (7.18).

From (7.18), due to the functional $w \to [A(y), w]_+$ is convex and lower semicontinuous on X (hence, it is continuous on X), we obtain that for any $w \in X$ $\langle d, w\rangle \leq [A(y), w]_+$. So, $d \in A(y)$.

The theorem is proved. □

In a standard way (see [26]), by using the results of the theorem 7.1, we can obtain such proposition.

Corollary 7.2. *Let $A : X \to C_v(X^*) \cap \mathscr{H}(X^*)$ be bounded map of the Volterra type that satisfies the property S_k on W_σ. Moreover, let for some $c > 0$*

$$\frac{[A(y), y]_+ - c\|A(y)\|_+}{\|y\|_X} \to +\infty \tag{7.25}$$

as $\|y\|_X \to +\infty$. *Then for any* $a \in H$, $f \in X^*$, *there exists at least one solution of the problem (7.3) that can be obtained by the Faedo–Galerkin method.*

Proof. Let us set $\varepsilon = \frac{\|a\|_H^2}{2c^2}$. We consider $w \in W$:

$$\begin{cases} w' + \varepsilon J(w) = \bar{0}, \\ w(0) = a, \end{cases}$$

where $J : X \to C_v(X^*)$ be the duality map. Hence $\|w\|_X \leq c$. We define $\hat{A} : X \to C_v(X^*) \bigcap \mathscr{H}(X^*)$ by the rule: $\hat{A}(z) = A(z+w)$, $z \in X$. Let us set $\hat{f} = f - w' \in X^*$. If $z \in W$ is the solution of the problem

$$\begin{cases} z' + \hat{A}(z) \ni f, \\ z(0) = \bar{0}, \end{cases}$$

then $y = z+w$ is the solution of the problem (7.3). It is clear that $\hat{A}$ is a bounded map of the Volterra type that satisfies the property S_k on W. So, due to the Theorem 7.1, it is enough to prove the $+$-coercivity for the map $\hat{A}$. This property follows from such estimates:

$$\begin{aligned} [\hat{A}(z), z]_+ &\geq [A(z+w), z+w]_+ - [A(z+w), w]_+ \\ &\geq [A(z+w), z+w]_+ - c\|A(z+w)\|_+, \\ \|z\|_X &\geq \|z+w\|_X - c, \end{aligned}$$

The corollary is proved. □

Analyzing the proof of the Theorem 7.1, we can obtain such result.

Corollary 7.3. *$A : X \to C_v(X^*) \cap \mathscr{H}(X^*)$ be bounded map of the Volterra type that satisfies the property S_k on W_σ, $\{a_n\}_{n\geq 0} \subset H$: $a_n \to a_0$ in H as $n \to +\infty$, $y_n \in W$, $n \geq 1$ be the corresponding to initial data a_n solution of the problem (7.3). If $y_n \rightharpoonup y_0$ in X, as $n \to +\infty$, then $y \in W$ is the solution of the problem (7.3) with initial data a_0. Moreover, up to a subsequence, $y_n \rightharpoonup y_0$ in $W_\sigma \bigcap C(S; H)$ (Fig. 7.5).*

Example 7.1. Let us consider the bounded domain $\Omega \subset \mathbf{R}^n$ with rather smooth boundary $\partial\Omega$, $S = [0, T]$, $Q = \Omega \times (0; T)$, $\Gamma_T = \partial\Omega \times (0; T)$. For $a, b \in \mathbf{R}$, we set $[a, b] = \{\alpha a + (1-\alpha)b | \alpha \in [0, 1]\}$. Let $V = H_0^1(\Omega)$ be real Sobolev space, $V^* = H^{-1}(\Omega)$ be its dual space, $H = L_2(\Omega)$, $a \in H$, $f \in X^*$. We consider such problem:

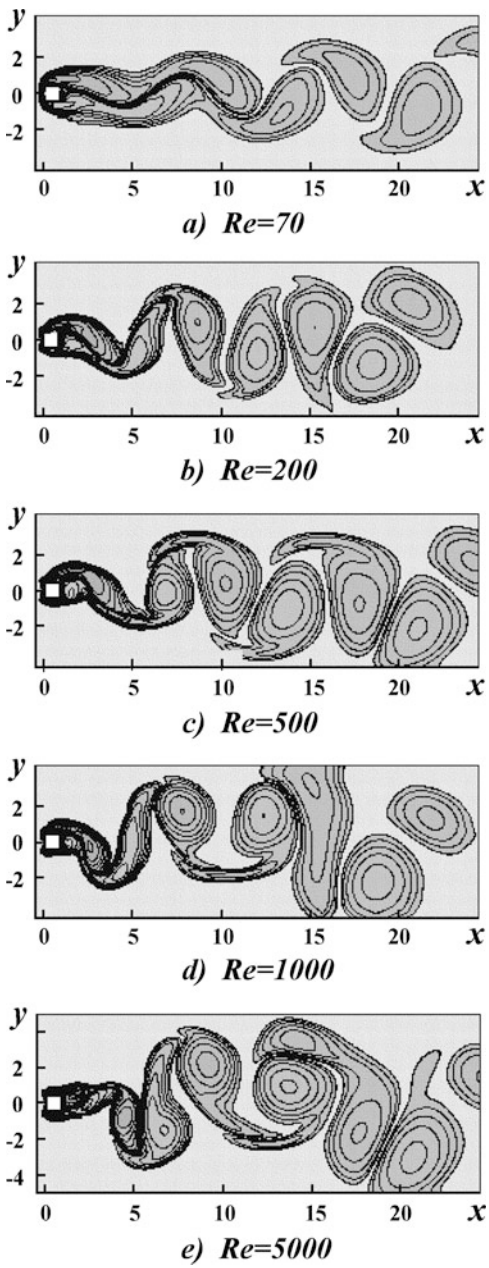

Fig. 7.5 Vorticity distribution in the wake past the square cylinder at different Reynolds numbers, $\tau = 25$ (see Sect. 6.4 and [41, Appendix A])

$$\frac{\partial y(x,t)}{\partial t} + [-\Delta y(x,t), \Delta y(x,t)] \ni f(x,t) \text{ in } Q,$$
$$y(x,0) = a(x) \text{ in } \Omega,$$
$$y(x,t) = 0 \text{ in } \Gamma_T. \tag{7.26}$$

We consider $A : X \to C_v(X^*) \cap \mathscr{K}(X^*)$,

$$A(y) = \{\Delta y \cdot p |\, p \in L_\infty(S), |\, p(t)| \leq 1 \text{ a.e. in } S\}.$$

where Δ means the energetic extension in X of Laplacian (see [11] for details), $(\Delta y \cdot p)(x,t) = \Delta y(x,t) \cdot p(t)$ for a.e. $(x,t) \in Q$.

We remark that

$$\|A(y)\|_+ = \|y\|_X,\ [A(y), y]_+ = \|y\|_X^2. \tag{7.27}$$

We rewrite the problem (7.26) to the next one (see [11] for details):

$$y' + A(y) \ni f,\ y(0) = a. \tag{7.28}$$

The solution of the problem (7.28) is called the generalized solution of (7.26). Due to the Corollary 7.2 and (7.27), it is enough to check that A satisfies the property S_k on W. Indeed, let $y_n \rightharpoonup y$ in W, $d_n \rightharpoonup d$ in X^*, where $d_n = p_n \Delta\, y_n$, $p_n \in L_\infty(S)$, $|p_n(t)| \leq 1$ for a.e. $t \in S$. Then $y_n \to y$ in Y and up to a subsequence $p_n \to p$ weakly star in $L_\infty(S)$, where $|p(t)| \leq 1$ for a.e. $t \in S$. As $\|p_n \Delta y_n - p_n \Delta y\|_{L_2(S;H^{-2}(\Omega))} \leq \|y_n - y\|_Y \to 0$, then $p_n \Delta\, y_n \to p \Delta\, y$ weakly in $L_2(S; H^{-2}(\Omega))$. Due to the continuous embedding $X^* \subset L_2(S; H^{-2}(\Omega))$, we obtain that $d = p\, \Delta\, y \in A(y)$. So, we obtain such statement.

Proposition 7.1. *Under the listed above conditions, the problem (7.26) has at least one generalized solution $y \in W$.*

7.2 Noncoercive Evolution Inclusions for S_k Type Operators

For a large class of noncoercive operator inclusions, including those generated by maps of S_k type, we obtain a general theorem on existence of solutions. We apply this result to a particular example. This theorem is proved using the method of Faedo–Galerkin approximations.

One of the most effective approaches to investigate nonlinear problems, represented by partial differential equations, inclusions, and inequalities with boundary values, consists in the reduction of them into differential-operator inclusions in infinite-dimensional spaces governed by nonlinear operators. In order to study these objects, the modern methods of nonlinear analysis have been used [7, 10, 11, 26]. Convergence of approximate solutions to an exact solution of the differential-operator equation or inclusion is frequently proved on the basis of the property of monotony or pseudomonotony of the corresponding operator. In applications, as an example of a pseudomonotone operator, the sum of a radially continuous monotone bounded operator and a strongly continuous operator was considered in [11]. Concrete examples of pseudomonotone operators were obtained by taking elliptic differential operators for which only the terms containing the highest derivatives satisfy the monotony property [26]. The papers [4, 5] became classical in the given direction of investigations. In particular, in the work [5], the class of generalized pseudomonotone operators was introduced. Let W be real Banach space continuously embedded in the real reflexive Banach space Y with dual space Y^*, and let $\langle\cdot,\cdot\rangle_Y : Y^* \times Y \to \mathbf{R}$ be the pairing between Y and Y^*. Further, by $C_v(Y^*)$, we denote the family of all nonempty closed convex bounded subsets of the space Y^*. The multivalued map $A : Y \to C_v(Y^*)$ is said to be *generalized pseudomonotone on W* if for each pair of sequences $\{y_n\}_{n\geq 1} \subset W$ and $\{d_n\}_{n\geq 1} \subset Y^*$ such that $d_n \in A(y_n)$, $y_n \to y$ weakly in W, $d_n \to d$ weakly in Y^*, from the inequality

$$\limsup_{n\to\infty} \langle d_n, y_n\rangle_Y \leq \langle d, y\rangle_Y,$$

it follows that $d \in A(y)$ and $\langle d_n, y_n\rangle_Y \to \langle d, y\rangle_Y$. I.V. Skrypnik's idea of passing to subsequences in classical definitions [39], which was used for stationary and evolution inclusions by several authors (see [17–20, 23, 24, 28, 29, 46, 47] and the citations there), enabled to consider the class of w_{λ_0}-pseudomonotone maps, which includes, in particular, the class of generalized pseudomonotone on W multivalued operators. Also, the sum of two w_{λ_0}-pseudomonotone maps continues to be w_{λ_0}-pseudomonotone. Let us remark that any multivalued map $A : Y \to C_v(Y^*)$ naturally generates the *upper* and, accordingly, *lower form*:

$$[\,A(y),\omega]_+ = \sup_{d\in A(y)} \langle d, w\rangle_Y,$$

$$[\,A(y),\omega]_- = \inf_{d\in A(y)} \langle d, w\rangle_Y, \qquad y,\omega \in X.$$

The properties of the given functionals have been investigated by M.Z. Zgurovsky and V.S. Mel'nik (see [24, 28, 46]). Thus, together with the classical coercivity condition for single-valued maps, that is,

$$\frac{\langle A(y), y\rangle_Y}{\|y\|_Y} \to +\infty, \qquad \text{as} \qquad \|y\|_Y \to +\infty,$$

which ensures some important a priori estimates, one can define the property of $+$-coercivity (and, accordingly, $-$-coercivity) for multivalued maps, that is,

$$\frac{[A(y), y]_{+(-)}}{\|y\|_Y} \to +\infty, \qquad \text{as} \qquad \|y\|_Y \to +\infty.$$

Clearly, $+$-coercivity is a weaker condition than $-$-coercivity.

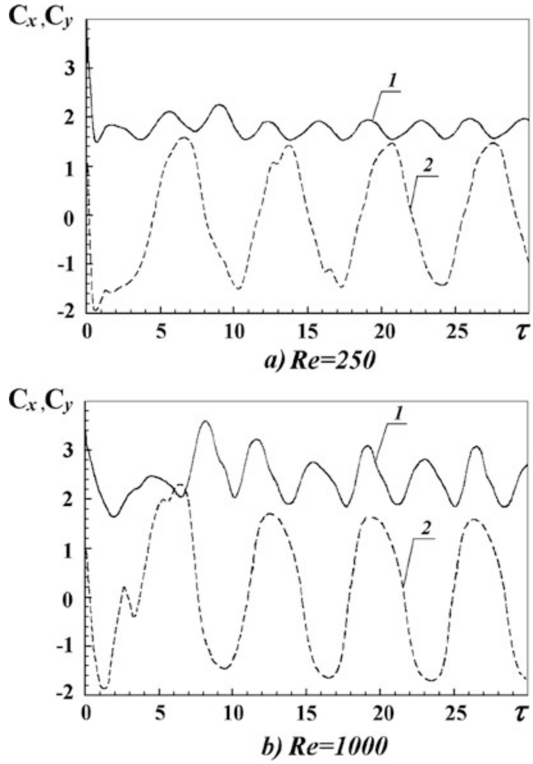

Fig. 7.6 Time dependence of drag coefficient C_x – **1** and of lifting force C_y – **2** of the square prism at different Reynolds numbers (see Sect. 6.4 and [41, Appendix A])

The recent development of the monotony method in the theory of differential-operator inclusions and evolutionary variational inequalities ensures existence of solutions of the associated equations if the operator is —-coercivity, quasibounded, and generalized pseudomonotone (see, e.g., [6, 9, 12–14, 32] and the citations there). Further, the results of V.S. Mel'nik [30] allowed to consider evolution inclusions with +-coercive w_{λ_0}-pseudomonotone quasibounded multimappings (see [17–20, 23, 24, 44, 48, 49] and the citations there) (Fig. 7.6).

In this chapter, we introduce a differential-operator scheme for the investigation of noncoercive nonlinear boundary-value problems for which the terms of the operator corresponding to the highest derivatives do not satisfy a monotony condition. In this framework, the properly S_k of a multivalued operator will be essential. In this way, we obtain a new theorem of existence of solutions for evolution inclusions.

A multivalued map $A : Y \to C_v(X^*)$ satisfies the *property* S_k *on* W, if for any sequence $\{y_n\}_{n\geq 0} \subset W$ such that $y_n \to y_0$ weakly in W, $d_n \to d_0$ weakly in Y^* as $n \to +\infty$, where $d_n \in A(y_n)$, $\forall n \geq 1$, from

$$\lim_{n\to\infty} \langle d_n, y_n - y_0\rangle_Y = 0,$$

it follows that $d_0 \in A(y_0)$.

Now, we consider the simple example of S_k type operator. Let $\Omega = (0, 1)$, $Y = H_0^1(\Omega)$ be the real Sobolev space with dual space $Y^* = H^{-1}(\Omega)$ (see for details [11]). Let $A : Y \times [-1, 1] \to Y^*$ be defined by the rule

$$A(y, \alpha) = -\frac{d}{dx}\left(\alpha \frac{d}{dx} y\right).$$

Then the multivalued map

$$\mathscr{A}(y) = \{A(y, \alpha) \,|\, \alpha \in [-1, 1]\}, \; y \in Y,$$

satisfies the property S_k; it is $+$-coercive, but not $-$-coercive; it is not generalized pseudomonotone and $(-\mathscr{A})$ is not generalized pseudomonotone too (see [15] for details). We remark that stationary inclusions for multimappings with the S_k property were considered by O.V. Kapustyan, P.O. Kasyanov, and O.P. Kogut [15], whereas evolution inclusions for $+$-coercive w_{λ_0}-pseudomonotone quasibounded maps were studied in [17–20, 23, 24] and [44, 48, 49].

We note that the results of this chapter are new for evolution equations too.

7.2.1 Preliminaries: On Some Classes of Multivalued Maps

In this section, we will consider some classes of multivalued maps, which are necessary in order to prove the theorem of existence of solutions for evolution inclusions.

Let Y be some reflexive Banach space, Y^* be its topologically adjoint, $\langle \cdot, \cdot \rangle_Y : Y^* \times Y \to \mathbf{R}$ be the pairing between them, and let $A : Y \rightrightarrows Y^*$ be a strict multivalued map, that is, $A(y) \neq \emptyset$, $\forall y \in X$. For this map, let us define the upper norm

$$\|A(y)\|_+ = \sup_{d \in \mathscr{A}(y)} \|d\|_{X^*}$$

and the lower norm

$$\|A(y)\|_- = \inf_{d \in \mathscr{A}(y)} \|d\|_{X^*},$$

where $y \in Y$. Let us consider also the maps $\mathrm{co}A : Y \rightrightarrows Y^*$ and $\overline{\mathrm{co}}A : Y \rightrightarrows Y^*$, which are defined by

$$(\mathrm{co}A)(y) = \mathrm{co}(A(y))$$

and

$$(\overline{\mathrm{co}}A(y)) = \overline{\mathrm{co}(A(y))},$$

respectively, where $\overline{\mathrm{co}(A(y))}$ is the weak closure of the convex hull of the set $A(y)$ in the space Y^*.

It is known that for any multivalued maps $A, B : Y \overset{\to}{\to} Y^*$, the following properties hold [24, 28, 47]:

1. $[A(y), v_1 + v_2]_+ \leq [A(y), v_1]_+ + [A(y), v_2]_+$.
2. $[A(y), v_1 + v_2]_- \geq [A(y), v_1]_- + [A(y), v_2]_-$, $\forall y, v_1, v_2 \in Y$.
3. $[A(y), v]_+ = -[A(y), -v]_-$.
4. $[A(y) + B(y), v]_{+(-)} = [A(y), v]_{+(-)} + [B(y), v]_{+(-)}$, $\forall y, v \in Y$.
5. $[A(y), v]_{+(-)} = [\overline{\mathrm{co}}A(y), v]_{+(-)}$, $\forall y, v \in Y$.
6. $[A(y), v]_{+(-)} \leq \|A(y)\|_{+(-)} \|v\|_Y$.
7. $\|A(y) + B(y)\|_+ \leq \|A(y)\|_+ + \|B(y)\|_+$.
8. $d \in \overline{\mathrm{co}}A(y)$ is true if and only if
$$[A(y), v]_+ \geq \langle d, v\rangle_Y, \quad \forall v \in Y.$$
9. Let $D \subset Y$. If $a(\cdot,\cdot) : D \times Y \to \mathbf{R}$, then for every $y \in D$, the functional $Y \ni w \mapsto a(y, w)$ is positively homogeneous convex and lower semicontinuous if and only if there exists a multivalued map $A : D \subset Y \to C_v(Y^*)$ with domain $D(A) = D$ such that
$$a(y, w) = [A(y), w]_+, \quad \forall y \in D(A),\ \forall w \in Y.$$

Further, $y_n \rightharpoonup y$ in Y will mean that y_n converges weakly to y in Y.

Let W be some normalized space continuously embedded into Y. Let us consider a multivalued map $A : Y \to C_v(Y^*)$. We shall introduce some important classes of maps.

Definition 7.4. A strict multi-valued map $A : Y \to C_v(Y^*)$ is called:

- λ_0-*pseudomonotone on* W, if for any sequence $\{y_n\}_{n\geq 0} \subset W$ such that $y_n \rightharpoonup y_0$ in W, $d_n \rightharpoonup d_0$ in Y^* as $n \to +\infty$, where $d_n \in A(y_n)$, $\forall n \geq 1$, from the inequality
$$\limsup_{n\to\infty} \langle d_n, y_n - y_0\rangle_Y \leq 0,$$
it follows the existence of a subsequence $\{y_{n_k}, d_{n_k}\}_{k\geq 1}$ from $\{y_n, d_n\}_{n\geq 1}$, for which
$$\liminf_{k\to\infty} \langle d_{n_k}, y_{n_k} - w\rangle_Y \geq [A(y_0), y_0 - w]_-,$$
for all $w \in Y$.
- A map satisfying the *property* S_k *on* W, if for any sequence $\{y_n\}_{n\geq 0} \subset W$ such that $y_n \to y_0$ weakly in W, $d_n \to d_0$ weakly in Y^* as $n \to +\infty$, where $d_n \in A(y_n)$, $\forall n \geq 1$, from
$$\lim_{n\to\infty} \langle d_n, y_n - y_0\rangle_Y = 0,$$
it follows that $d_0 \in A(y_0)$.
- *Bounded*, if for every $L > 0$, there exists $l > 0$, such that for any $y \in Y$ satisfying $\|y\|_Y \leq L$ it follows that $\|A(y)\|_+ \leq l$.

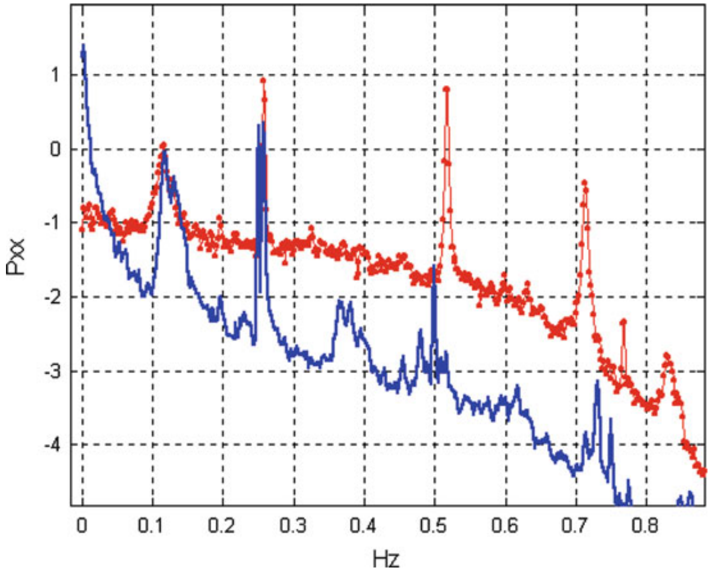

Fig. 7.7 Dependence of a Strouhal number on a Reynolds number for the flow past the square prism and comparison of the obtained results with the experimental data. [33]: *black circle* computations, *open circle*, *open triangle* ... experiment (see Sect. 6.4 and [41, Appendix A])

Definition 7.5. A strict multivalued map $A : X \to C_v(X^*)$ is called:

- *An operator of the Volterra type* , if for arbitrary $u, v \in X, t \in S$ from the equality $u(s) = v(s)$ for a.e. $s \in [0, t]$, it follows that $[A(u), \xi_t]_+ = [A(v), \xi_t]_+$, $\forall \xi_t \in X$ such that $\xi_t(s) = 0$ for a.e. $s \in S \setminus [0, t]$.
- *+(−)-coercive*, if there exists a real function $\gamma : \mathbf{R}_+ \to \mathbf{R}$ such that $\gamma(s) \to +\infty$ as $s \to +\infty$ and

$$[A(y), y]_{+(-)} \geq \gamma(\|y\|_Y)\|y\|_Y, \quad \forall y \in Y.$$

- *Demiclosed*, if from the fact that $y_n \to y$ in Y, $d_n \rightharpoonup d$ in Y^*, where $d_n \in A(y_n)$, $n \geq 1$, it follows that $d \in A(y)$ (Fig. 7.7).

7.2.2 Setting of the Problem

Let V, H be real Hilbert spaces. We consider the inner product $(\cdot, \cdot)$ in H and identify this space with its conjugate H^*. Let the embedding $V \subset H$ be compact. Then we obtain the following chain of compact and dense embeddings:

$$V \subset H \subset V^*,$$

where V^* is the topologically adjoint space to V. Let us denote a finite interval of time by $S = [0, T]$, and let

$$X = L_2(S; V), \quad X^* = L_2(S; V^*),$$
$$Y \equiv Y^* = L_2(S; H).$$

The linear space $W = \{y \in X \mid y' \in X^*\}$ is a Hilbert space with the norm $\|y\|_W = \|y\|_X + \|y'\|_{X^*}$, where y' is the derivative of $y \in X$ in the sense of the space of distributions $\mathscr{D}^*(S; V^*)$ [11]. For an arbitrary $v \in X$ and $f \in X^*$, let us consider

$$\langle f, v \rangle = \int_S \langle f(\tau), v(\tau) \rangle_V \, d\tau.$$

Here, $\langle \cdot, \cdot \rangle_V : V^* \times V \to \mathbf{R}$ is the canonical pairing, which coincides with the inner product $(\cdot, \cdot)$ in H on $H \times V$. Hence, $\langle f, v \rangle = \int_S (f(\tau), v(\tau))\, d\tau$ if $f \in Y$. In the sequel, for simplicity, we shall use the last notation even if $f \in X^*$.

For a multivalued (in the general case) map $A : X \rightrightarrows X^*$, let us consider the problem

$$\begin{cases} u' + A(u) \ni f, \\ u(0) = a, \ u \in W \subset C(S; H), \end{cases} \tag{7.29}$$

where $a \in H$ and $f \in X^*$ are arbitrary fixed elements. The goal of this work is to prove the solvability for the given problem by the Faedo–Galerkin method.

Let us introduce a class of subsets of X^*, which will be denoted by $\mathscr{H}(X^*)$. We shall say that $B \in \mathscr{H}(X^*)$, if for an arbitrary measurable set $E \subset S$ and for arbitrary $u, v \in B$, the inclusion $u + (v - u)\chi_E \in B$ is true. Here and further for $d \in X^*$ $(d\chi_E)(\tau) = d(\tau)\chi_E(\tau)$ for a.e. $\tau \in S$, where

$$\chi_E(\tau) = \begin{cases} 1, \ \tau \in E, \\ 0, \text{ else.} \end{cases}$$

Lemma 7.3. *$B \in \mathscr{H}(X^*)$ if and only if for all $n \geq 1$, $\{d_i\}_{i=1}^n \subset B$, and for arbitrary measurable pairwise disjoint subsets $\{E_j\}_{j=1}^n$ of the set S such that $\cup_{j=1}^n E_j = S$, it holds $\sum_{j=1}^n d_j \chi_{E_j} \in B$.*

Let us remark that $\emptyset, X^* \in \mathscr{H}(X^*)$ and that the for all $f \in X^*$, the constant function $h(t) \equiv f$ belongs to $\mathscr{H}(X^*)$. On the other hand, if $K : S \rightrightarrows V^*$ is an arbitrary multivalued map, then

$$\{f \in X^* \mid f(t) \in K(t) \text{ for a.e. } t \in S\} \in \mathscr{H}(X^*).$$

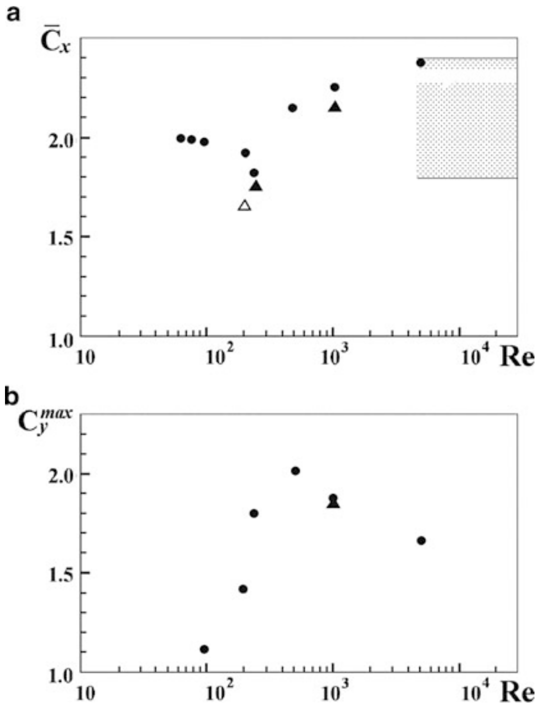

Fig. 7.8 Dependencies of period average drag coefficient $\overline{C}_x$ – ***a*** and the lifting force coefficient oscillation amplitude C_y^{max} – ***b*** of the square prism on Reynolds numbers Computation: *black circle* – [41, Appendix A], *black triangle* – [8], *open triangle* – [31] Experiment: by the results from [3, 41]. (see Sect. 6.4 and [41, Appendix A])

At the same time, for an arbitrary $v \in V^* \setminus \bar{0}$ (which is not equal to 0), the closed convex set $B = \{f \in X^* \mid f \equiv \alpha v,\ \alpha \in [0,1]\} \notin \mathscr{H}(X^*)$, as $g(\cdot) = v \cdot \chi_{[0,T/2]}(\cdot) \notin B$.

In order to prove the solvability for problem (7.29), we will obtain some auxiliary statements.

Let us consider a complete system of vectors $\{h_i\}_{i \geq 1} \subset V$ such that:

(α1) $\{h_i\}_{i \geq 1}$ is orthonormal in H.
(α2) $\{h_i\}_{i \geq 1}$ is orthogonal in V.
(α3) $(h_i, v)_V = \lambda_i (h_i, v), \quad \forall v \in V, \forall\, i \geq 1,$
where $0 \leq \lambda_1 \leq \lambda_2, \ldots, \lambda_j \longrightarrow \infty$ as $j \longrightarrow \infty$, and $(\cdot,\cdot)_V$ is the natural inner product in V.

Thus, $\{h_i\}_{i \geq 1}$ is a special basis [40]. Let $H_m = \text{span}\{h_i\}_{i=1}^m$, for each $m \geq 1$, on which we consider the inner product induced from H, denoted by $(\cdot,\cdot)$. Due to the equivalence of H^* and H, it follows that $H_m^* \equiv H_m$, $X_m = L_2(S; H_m) \equiv X_m^*$, and $\langle \cdot,\cdot \rangle_{X_m} = \langle \cdot,\cdot \rangle$. We put $W_m := \{y \in X_m \mid y' \in X_m^*\}$, where y' is the derivative of an element $y \in X_m$ considered in the sense of $\mathscr{D}^*(S, H_m)$. Let us remark that the embeddings $X_m \subset Y_m \subset X_m^*$ are continuous and that the embedding W_m into X_m is compact [26, p.70] (Fig. 7.8).

Let further $I : X \to X^*$ be the canonical embedding.

Let us fix $\lambda \in \mathbf{R}$. Let us set $\varphi_\lambda(t) = e^{-\lambda t}$, $t \in S$. For an arbitrary $y \in X^*$, let us define y_λ (as a map from S into V^*) in the following way: $y_\lambda(t) = \varphi_\lambda(t) y(t)$

for a.e. $t \in S$. Let us remark that $(y_\lambda)_{-\lambda} = y$, for all $y \in X^*$. Also, we define the element $\varphi_\lambda y$ by $(\varphi_\lambda y)(t) = y(t)\varphi_\lambda(t)$ for a.e. $t \in S$.

Lemma 7.4. *The map $y \mapsto y_\lambda$ is an isomorphism and an homeomorphism as a map acting from X_m into X_m (respectively from X_m^* into X_m^*, from W_m into W_m, from X into X, from X^* into X^*, from Y_m into Y_m, from Y into Y). Moreover, the map $W_m \ni y \longmapsto y_\lambda \in W_m$ is weakly-weakly continuous, that is, from the fact that $y_n \rightharpoonup y$ in W_m, it follows that $y_{n,\lambda} \rightharpoonup y_\lambda$ in W_m. Also, we have $y'_\lambda = \varphi'_\lambda y + \varphi_\lambda y' \in X_m^*$, $\forall y \in W_m$.*

Let us consider the multivalued map $\mathscr{A} : X \to C_v(X^*)$. Let us define the set $\mathscr{A}_\lambda(y_\lambda) \in C_v(X^*)$ for fixed $y \in X$ by the next relation

$$[\mathscr{A}_\lambda(y_\lambda), \omega]_+ = [\mathscr{A}(y) + \lambda y, \omega_\lambda]_+, \quad \forall \omega \in X.$$

Let us remark that as the functional $\omega \longmapsto [\mathscr{A}(y) + \lambda y, \omega_\lambda]_+$ is semiadditive, positively homogeneous, and lower semicontinuous (as the supremum of linear and continuous functionals), $\mathscr{A}_\lambda(y_\lambda)$ is defined correctly.

Lemma 7.5. *If the map $\mathscr{A} : X \to C_v(X^*)$ is bounded, then $\mathscr{A}_\lambda : X \to C_v(X^*)$ is bounded.*

Let us consider now multivalued maps which map X_m into X_m^*.

Definition 7.6. The multivalued map $\mathscr{A} : X_m \to C_v(X_m^*)$ is called (W_m, X_m^*)-weakly closed, if from that fact that $y_n \rightharpoonup y$ in W_m, $d_n \rightharpoonup d$ in X_m^*, $d_n \in \mathscr{A}(y_n)$, $\forall n \geq 1$, it follows that $d \in \mathscr{A}(y)$.

Lemma 7.6. *The multivalued map $\mathscr{A} : X_m \to C_v(X_m^*)$ satisfies the property S_k on W_m if and only if $\mathscr{A} : X_m \to C_v(X_m^*)$ is (W_m, X_m^*)-weakly closed.*

Proof. Let us prove the necessity. Let $y_n \rightharpoonup y$ in W_m, $d_n \rightharpoonup d$ in X_m^*, where $d_n \in \mathscr{A}(y_n)$, $\forall n \geq 1$. Then $y_n \to y$ in X_m and $\langle d_n, y_n - y\rangle_{X_m} \to 0$ as $n \to +\infty$. Therefore, since $\mathscr{A}$ satisfies the S_k property on W_m, we obtain that $d \in \mathscr{A}(y)$.

Let us prove the sufficiency. Let $y_n \rightharpoonup y$ in W_m, $d_n \rightharpoonup d$ in X_m^*, $\langle d_n, y_n - y\rangle_{X_m} \to 0$ as $n \to +\infty$, where $d_n \in \mathscr{A}(y_n)$, $\forall n \geq 1$. Then $y_n \to y$ in X_m and $d \in \mathscr{A}(y)$. The lemma is proved. □

Lemma 7.7. *If $\mathscr{A} : X_m \to C_v(X_m^*)$ is (W_m, X_m^*)-weakly closed, then $\mathscr{A}_\lambda$ is (W_m, X_m^*)-weakly closed.*

Proof. Let $y_{n,\lambda} \rightharpoonup y_\lambda$ in W_m, $d_n \rightharpoonup d$ in X_m^*, $d_n \in \mathscr{A}_\lambda(y_{n,\lambda})$. Then in virtue of Lemma 7.4 $y_n := (y_{n,\lambda})_{-\lambda} \rightharpoonup y := (y_\lambda)_{-\lambda}$ in W_m, $y_{n,\lambda} \to y_\lambda$ in X_m and $y_n \to y$ in X_m.

Further, as $[\mathscr{A}(y_n)+\lambda y_n, \omega_\lambda]_+ \geq \langle d_n, \omega\rangle_{X_m}$, for any $\omega \in X_m$, we obtain $d_{n,-\lambda} \in \mathscr{A}(y_n) + \lambda y_n$. Therefore, $g_n := d_{n,-\lambda} - \lambda y_n \in \mathscr{A}(y_n)$. Let us remark that $d_{n,-\lambda} = (d_n)_{-\lambda} \rightharpoonup d_{-\lambda}$ in X_m^*, and since $X_m \subset X_m^*$ continuously, we have $g_n \rightharpoonup g$ in X_m^*

for some $g \in X_m^*$. Due to the fact that $\mathscr{A}$ is (W_m, X_m^*)-weakly closed, we have that $g \in \mathscr{A}(y)$. Therefore, $d_{n,-\lambda} - \lambda y_n \rightharpoonup g$ in X_m^*, so that $d_{n,-\lambda} \rightharpoonup \lambda y + g$ in X_m^*, and then, $d_n = (d_{n,-\lambda})_\lambda \rightharpoonup \lambda y_\lambda + g_\lambda$ in X_m^*. Therefore, for all $\omega \in X_m$, we get

$$\begin{aligned}\langle d, \omega\rangle_{X_m} &= \langle \lambda y_\lambda + g_\lambda, \omega\rangle_{X_m} \\ &= \langle \lambda y + g, \omega_\lambda\rangle_{X_m} \le [\mathscr{A}(y) + \lambda y, \omega_\lambda]_+,\end{aligned}$$

so

$$d \in \mathscr{A}_\lambda(y_\lambda).$$

The lemma is proved. □

Since the embedding W_m into X_m is compact, from Lemmas 7.6, 7.7, it follows the following corollary.

Corollary 7.4. *If the multivalued map $\mathscr{A} : X_m \to C_v(X_m^*)$ satisfies the property S_k on W_m, then $\mathscr{A}_\lambda$ is λ_0-pseudomonotone on W_m.*

7.2.3 Main Results

In the next theorem, using the Faedo–Galerkin method, we will prove the existence of solutions for problem (7.29).

Theorem 7.2. *Let $a = \bar{0}$, $A : X \to C_v(X^*) \cap \mathscr{H}(X^*)$ be a bounded map of the Volterra type, which satisfies the property S_k on W. Moreover, let for some $\lambda \ge 0$ the map $A + \lambda I$ be $+$-coercive. Then for arbitrary $f \in X^*$, there exists at least one solution of problem (7.29), which can be obtained by the Faedo–Galerkin method.*

Proof. We shall divide the proof in several steps.
Step 1: A Preliminary Estimate.

At first, let us show that there exists a real nondecreasing function $\gamma : \mathbf{R}_+ \to \mathbf{R}$ such that $\gamma(r) \to +\infty$ as $r \to +\infty$, it is bounded from below on bounded sets and for any $y \in X$,

$$\begin{aligned}\sup_{d \in A(y)} \int_0^T e^{-2\lambda\tau}(d(\tau) + \lambda y(\tau), y(\tau))d\tau \\ \ge \gamma(\|y\|_X)\|y\|_X. \qquad (7.30)\end{aligned}$$

For an arbitrary $r > 0$, let us set

$$\widetilde{\gamma}(r) = \inf_{y \in X, \|y\|_X = r} \sup_{d \in A(y)} \frac{\langle d + \lambda y, y\rangle_X}{\|y\|_X}$$

and $\widetilde{\gamma}(0) := 0$. The following properties hold:

(a) As A is bounded and the embedding $X \subset X^*$ is continuous, we have $\widetilde{\gamma}(r) > -\infty$.
(b) From the construction of the function $\widetilde{\gamma}$, we have that for all $y \in X$,

$$[A(y) + \lambda y, y]_+ \geq \widetilde{\gamma}(\|y\|_X)\|y\|_X. \tag{7.31}$$

In virtue of the boundedness of A, it follows that $\widetilde{\gamma}$ is bounded from below on bounded sets.
(c) From the $+$-coercivity of $A + \lambda I$, it follows that $\widetilde{\gamma}(r) \to +\infty$ as $r \to +\infty$.
(d) From (a)–(c), we have $\inf\limits_{r \geq 0} \widetilde{\gamma}(r) =: a > -\infty$.

For an arbitrary $b > a$, let us consider the nonempty bounded set of $\mathbf{R}_+$ given by $A_b = \{c \geq 0 \mid \widetilde{\gamma}(c) \leq b\}$. Let $c_b = \sup_{c \in A_b} c$, $b > a$. Let us remark that $c_{b_2} \leq c_{b_1} < +\infty$, for all $b_1 > b_2 > a$, and $c_b \to +\infty$ as $b \to +\infty$. Let us set

$$\widehat{\gamma}(t) = \begin{cases} a, & t \in [0, c_{a+1}], \\ a + k, & t \in (c_{a+k}, c_{a+k+1}],\ k \geq 1. \end{cases}$$

Then $\widehat{\gamma} : \mathbf{R}_+ \to \mathbf{R}$ is a bounded from below function on bounded sets of $\mathbf{R}_+$, it is a nondecreasing function such that $\widehat{\gamma}(r) \to +\infty$, as $r \to \infty$, and $\widehat{\gamma}(t) \leq \widetilde{\gamma}(t)$, for any $t \geq 0$ (Fig. 7.9).

Let us fix an arbitrary $y \in X$. Since A is the operator of the Volterra type, for all $t \in S$, we have

$$\sup_{d \in A(y)} \int_0^t \big(d(\tau) + \lambda y(\tau), y(\tau)\big) d\tau = \sup_{d \in A(y)} \int_0^T \big(d(\tau) + \lambda y_t(\tau), y_t(\tau)\big) d\tau$$

$$\geq \widehat{\gamma}(\|y_t\|_X)\|y_t\|_X = \widehat{\gamma}(\|y\|_{X_t})\|y\|_{X_t},$$

where $\|y\|_{X_t} = \|y_t\|_X$, $y_t(\tau) = \begin{cases} y(\tau), & \tau \in [0, t], \\ \bar{0}, & \text{else}. \end{cases}$ Let for an arbitrary $d \in A(y)$

$$g_d(\tau) = \big(d(\tau) + \lambda y(\tau), y(\tau)\big), \quad \text{for a.e.} \quad \tau \in S,$$

$$h(t) = \widehat{\gamma}(\|y\|_{X_t})\|y\|_{X_t}, \quad t \in S.$$

Let us remark that $h(t) \geq \min\{\widehat{\gamma}(0), 0\}\|y\|_X$ and

$$\sup_{d \in A(y)} \int_0^t g_d(\tau) d\tau \geq h(t),\ t \in S.$$

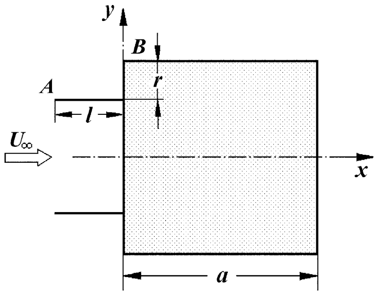

Fig. 7.9 Configuration of the body with control plates (see Sect. 6.4 and [41, Appendix A])

Let us show that

$$\sup_{d\in A(y)}\int_0^T e^{-2\lambda\tau}(d(\tau)+\lambda y(\tau),y(\tau))d\tau \geq e^{-2\lambda T}\sup_{d\in A(y)}\int_0^T (d(\tau)$$
$$+\lambda y(\tau),y(\tau))d\tau \sup_{d\in A(y)}\int_0^T (e^{-2\lambda\tau}-e^{-2\lambda T})$$
$$\times (d(\tau)+\lambda y(\tau),y(\tau))d\tau. \tag{7.32}$$

Let us set $\varphi(\tau) = e^{-2\lambda(T-\tau)}$, $\tau \in [0,T]$ (so $\varphi \in (0,1]$). For any $n \geq 1$ we put $\varphi_n(\tau) = \sum_{i=0}^{n-1}\varphi(\frac{iT}{n})\chi_{[\frac{iT}{n},\frac{(i+1)T}{n})}(\tau)$, $\tau \in [0,T]$. Then $\varphi(\frac{iT}{n})d_1 + (1-\varphi(\frac{iT}{n}))d_2 \in A(y)$, $\forall d_1 \in A(y)$, $\forall d_2 \in A(y)$, $\forall i = \overline{0, n-1}$. Let us remark that $|\varphi_n(\tau)-\varphi(\tau)| \leq \frac{2\lambda T}{n}$, $\forall \tau \in [0,T]$. In virtue of Lemma 7.3, we will obtain that $d = \sum_{i=0}^{n-1}(\varphi(\frac{iT}{n})d_1 + (1-\varphi(\frac{iT}{n}))d_2)\chi_{[t_i,t_{i+1})}(\tau) \in A(y)$, where $t_i = \frac{iT}{n}$. Therefore,

$$\sup_{d\in A(y)}\int_0^T e^{-2\lambda\tau}(d(\tau)+\lambda y(\tau),y(\tau))d\tau \geq \int_0^T (d(\tau)+\lambda y(\tau),y(\tau)e^{-2\lambda\tau})d\tau$$
$$= \int_0^T \varphi_n(\tau)(d_1(\tau)+\lambda y(\tau),y(\tau)e^{-2\lambda\tau})d\tau$$
$$+\int_0^T (1-\varphi_n(\tau))(d_2(\tau)$$
$$+\lambda y(\tau),y(\tau)e^{-2\lambda\tau})d\tau$$

$$\geq e^{-2\lambda T} \int_0^T (d_1(\tau) + \lambda y(\tau), y(\tau))d\tau$$
$$+ \int_0^T (e^{-2\lambda\tau} - e^{-2\lambda T})(d_2(\tau)$$
$$+ \lambda y(\tau), y(\tau))d\tau$$
$$- \frac{4\lambda T}{n}\big(\|A(y)\|_+ \|y\|_X + \lambda\|y\|_Y^2\big).$$

If $n \to +\infty$, then taking the supremum with respect to $d_1 \in A(y)$ and $d_2 \in A(y)$ in the last inequality, we will obtain (7.32). From (7.32), it follows that

$$\sup_{d\in A(y)} \int_0^T e^{-2\lambda\tau}(d(\tau) + \lambda y(\tau), y(\tau))d\tau$$
$$\geq e^{-2\lambda T} h(T) + 2\lambda \sup_{d\in A(y)} \int_0^T e^{-2\lambda s} \int_0^s g_d(\tau)d\tau ds$$
$$\geq e^{-2\lambda T} h(T)$$
$$+ 2\lambda T \sup_{d\in A(y)} \inf_{s\in S} e^{-2\lambda s} \int_0^s (d(\tau) + \lambda y(\tau), y(\tau))d\tau.$$

Let us show that

$$\sup_{d\in A(y)} \inf_{s\in S} e^{-2\lambda s} \int_0^s (d(\tau) + \lambda y(\tau), y(\tau))d\tau \geq -c_1\|y\|_X,$$

where $c_1 = \max\{-\widehat{\gamma}(0), 0\} \geq 0$ does not depend on $y \in X$. Let $y \in X$ is fixed. For $s \in S$, $d \in A(y)$ let us set

$$\varphi(s,d) = e^{-2\lambda s} \int_0^s \big(d(\tau) + \lambda y(\tau), y(\tau)\big)d\tau,$$
$$a = \sup_{d\in A(y)} \inf_{s\in S} \varphi(s,d), \;\; S_d = \{s \in S \mid \varphi(s,d) \leq a\}.$$

From the continuity of $\varphi(\cdot, d)$ on S, it follows that S_d is a nonempty closed set for an arbitrary $d \in A(y)$. Indeed, for any fixed $d \in A(y)$, there exists $s_d \in S$ such that

$$\varphi(s_d, d) = \min_{\hat{s}\in S} \varphi(\hat{s}, d) \le a.$$

From the continuity of $\varphi(\cdot, d)$ on S, it follows that S_d is closed.

Let us prove now that the system $\{S_d\}_{d\in A(y)}$ is centered. For fixed $\{d_i\}_{i=1}^n \subset A(y)$, $n \ge 1$, let us set

$$\begin{aligned}
\psi_i(\cdot) &= \big(d_i(\cdot) + \lambda y(\cdot), y(\cdot)\big),\\
\psi(\cdot) &= \max_{i\in\{1,\dots,n\}} \psi_i(\cdot),\\
E_0 &= \emptyset,\\
E_j &= \left\{\tau \in S \setminus \left(\cup_{i=0}^{j-1} E_i\right) \mid \psi_j(\tau) = \psi(\tau)\right\},
\end{aligned}$$

for $j = \overline{1,n}$, and

$$d(\cdot) = \sum_{j=1}^{n} d_j(\cdot)\chi_{E_j}(\cdot).$$

Let us remark that E_j is measurable for any $j = \overline{1,n}$, $\cup_{j=1}^n E_j = S$, $E_i \cap E_j = \emptyset$, $\forall i \ne j, i, j = \overline{1,n}$. Also, $d \in X^*$. Moreover,

$$\begin{aligned}
\varphi(s, d_i) = e^{-2\lambda s}\int_0^s \psi_i(\tau)d\tau &\le e^{-2\lambda s}\int_0^s \psi(\tau)d\tau\\
&= \varphi(s, d), \quad s \in S,\ i = \overline{1,n}.
\end{aligned}$$

Therefore, in virtue of Lemma 7.3, we have $d \in A(y)$ and for some $s_d \in S$,

$$\varphi(s_d, d_i) \le \varphi(s_d, d) = \min_{\hat{s}\in S} \varphi(\hat{s}, d) \le a, \quad i = \overline{1,n}.$$

So, $s_d \in \cap_{i=1}^n S_{d_i} \ne \emptyset$.

Since S is compact, and the system of closed sets $\{S_d\}_{d\in A(y)}$ is centered, we obtain the existence of $s_0 \in S$ such that $s_0 \in \cap_{d\in A(y)} S_d$. This implies that

$$\begin{aligned}
&\sup_{d\in A(y)} \inf_{s\in S} e^{-2\lambda s}\int_0^s \big(d(\tau) + \lambda y(\tau), y(\tau)\big)d\tau\\
&\quad\ge \sup_{d\in A(y)} e^{-2\lambda s_0}\int_0^{s_0} \big(d(\tau) + \lambda y(\tau), y(\tau)\big)d\tau
\end{aligned}$$

$$= e^{-2\lambda s_0} \sup_{d\in A(y)} \int_0^{s_0} g_d(\tau)d\tau \geq e^{-2\lambda s_0} h(s_0)$$
$$\geq e^{-2\lambda s_0} \min\{\widehat{\gamma}(0), 0\}\|y\|_X$$
$$\geq -\max\{-\widehat{\gamma}(0), 0\}\|y\|_X = -c_1\|y\|_X.$$

So, for all $y \in X$,

$$\sup_{d\in A(y)} \int_0^T e^{-2\lambda\tau}(d(\tau) + \lambda y(\tau), y(\tau))d\tau \geq \big(e^{-2\lambda T}\widehat{\gamma}(\|y\|_X) - 2\lambda c_1 T\big)\|y\|_X.$$

If we set $\gamma(r) = e^{-2\lambda T}\widehat{\gamma}(r) - 2\lambda c_1 T$, then we will obtain (7.30).

From (7.30), the properties of the real function γ and the conditions of the theorem, it follows the existence of $r_0 > 0$ such that $\gamma(r_0) > \|f_\lambda\|_{X^*} \geq 0$ and also that for any $y \in X$,

$$[A_\lambda(y_\lambda), y_\lambda]_+ \geq \gamma(\|y\|_X)\|y\|_X \geq \gamma(\|y_\lambda\|_X)\|y_\lambda\|_X.$$

Therefore, for all $y \in X$ satisfying $\|y_\lambda\|_X = r_0$, we have

$$[A_\lambda(y_\lambda) - f_\lambda, y_\lambda]_+ \geq (\gamma(r_0) - \|f_\lambda\|_{X^*})r_0 \geq 0,$$

that is,

$$[A_\lambda(y_\lambda) - f_\lambda, y_\lambda]_+ \geq 0,. \tag{7.33}$$

Step 2: Finite-Dimensional Approximations

We shall consider now a sequence of finite-dimensional approximative problems by the Faedo–Galerkin method.

For any $m \geq 1$, let $I_m \in \mathscr{L}(X_m; X)$ be the canonical embedding of X_m into X, and I_m^* be the adjoint operator to I_m. Then

$$\|I_m^*\|_{\mathscr{L}(X^*; X^*)} = 1,\ \forall m \geq 1. \tag{7.34}$$

Let us consider the following maps [17]:

$$A_m := I_m^* \circ A \circ I_m : X_m \to C_v(X^*),$$
$$A_{\lambda,m} := I_m^* \circ A_\lambda \circ I_m : X_m \to C_v(X^*),$$
$$A_{m,\lambda} := (A_m)_\lambda : X_m \to C_v(X^*),$$
$$f_m := I_m^* f,\ f_{\lambda,m} := I_m^* f_\lambda,\ f_{m,\lambda} := (f_m)_\lambda.$$

Let us remark that

$$A_{\lambda,m} = A_{m,\lambda}, \;\; f_{\lambda,m} = f_{m,\lambda}. \tag{7.35}$$

Indeed, in virtue of Lemma 7.4 for any $y, w \in X_m$,

$$\begin{aligned}[A_{\lambda,m}(y_\lambda), w]_+ &= [(I_m^* \circ A_\lambda)(y_\lambda), w]_+ = [A_\lambda(y_\lambda), w]_+ \\ &= [A(y) + \lambda y, w_\lambda]_+ = [I_m^* \circ (A + \lambda I)(y), w_\lambda]_+ \\ &= [(A_m)_\lambda(y_\lambda), w]_+ = [A_{m,\lambda}(y_\lambda), w]_+.\end{aligned}$$

So, from (7.33), (7.35), Lemma 7.5, Corollary 7.4, and the conditions of the theorem, applying similar arguments as in [17, pp. 115–117], [20, pp. 197–198], we will obtain the following properties:

(j1) $A_{\lambda,m}$ is λ_0-pseudo-monotone on W_m.
(j2) $A_{\lambda,m}$ is bounded.
(j3) $[A_{\lambda,m}(y_\lambda) - f_{\lambda,m}, y_\lambda]_+ \geq 0$, for all $y_\lambda \in X_m$ such that $\|y_\lambda\|_X = r_0$.

We note that (j3) is a consequence of (7.33) and the definition of $A_{\lambda,m}$, $f_{\lambda,m}$, whereas (j2) follows from Lemma 7.5 and the boundednesss of I_m, I_m^*. Finally, (j1) is obtained in the following way: since A satisfies the property S_k in W, for A_m, the same property holds on W_m; hence, by Corollary 7.4, the operator $A_{m,\lambda} = (A_m)_\lambda$ is λ_0-pseudomonotone in W_m, and then, (7.35) implies (j1).

Let us consider the operator $L_m : D(L_m) \subset X_m \to X_m^*$ with domain

$$D(L_m) = \{y \in W_m \mid y(0) = \bar{0}\} = W_m^0,$$

which is defined by the rule: $L_m y = y'$, $\forall y \in W_m^0$, where the derivative y' we consider in the sense of the space of distributions $\mathscr{D}^*(S; H_m)$. From [17, Lemma 5, p.117] for the operator L_m, the next properties are true:

(j4) L_m is linear.
(j5) $\langle L_m y, y\rangle \geq 0$, $\forall y \in W_m^0$.
(j6) L_m is maximal monotone.

Therefore, conditions (j_1)–(j_6) and Theorem 3.1 from [18] guarantee the existence of at least one solution $z_m \in D(L_m)$ of the problem:

$$L_m(z_m) + A_{\lambda,m}(z_m) \ni f_{\lambda,m}, \quad \|z_m\|_X \leq r_0,$$

which can be obtained by the method of singular perturbations. This means (see (7.35)) that $y_m := (z_m)_{-\lambda} \in W_m$ is the solution of the problem

$$\begin{cases} y_m' + A_m(y_m) \ni f_m \\ y_m(0) = \bar{0}, \;\; y_m \in W_m, \;\; \|y_m\|_X \leq R, \end{cases} \tag{7.36}$$

where $R = r_0 e^{\lambda T}$.

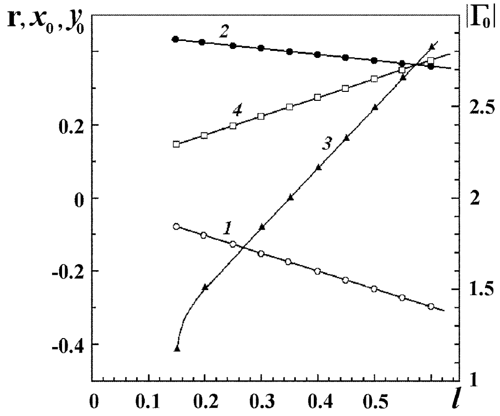

Fig. 7.10 Dependencies of standing vortex circulation, its disposition, and the parameter r on the plate length l: ***1*** – $x_0(l)$, ***2*** – $y_0(l)$, ***3*** – $\Gamma_0(l)$, ***4*** – $r(l)$ (see Sect. 6.4 and [41, Appendix A])

Step 3: Passing to the Limit

From (7.36), it follows that for any $m \geq 1$, there exists $d_m \in A(y_m)$ such that

$$I_m^* d_m = f_m - y_m' \in A_m(y_m) = I_m^* A(y_m). \tag{7.37}$$

Let us prove now that (up to a subsequence) the sequence of solutions of (7.36) converges to a solution of (7.29). Again, we divide this proof in some substeps.

Step 1a.

The boundedness of A and (7.36) implies that $\{d_m\}_{m\geq 1}$ is bounded in X^*. Therefore, there exists $c_1 > 0$ such that

$$\|d_m\|_{X^*} \leq c_1, \forall m \geq 1. \tag{7.38}$$

Step 3b.

Let us prove the boundedness of $\{y_m'\}_{m\geq 1}$ in X^*. From (7.37), it follows that $y_m' = I_m^*(f - d_m)$, $\forall m \geq 1$, and taking into account (7.34), (7.36) and (7.38), we have

$$\|y_m'\|_{X^*} \leq \|y_m\|_W \leq R + \|f\|_{X^*} + c_1 =: c_2. \tag{7.39}$$

In virtue of the continuous embedding $W \subset C(S; H)$, we obtain the existence of $c_3 > 0$ such that

$$\|y_m(t)\|_H \leq c_3, \ \forall m \geq 1, \ \forall t \in S. \tag{7.40}$$

Step 3c.

In virtue of estimates (7.38)–(7.40), due to the Banach–Alaoglu theorem, and taking into account the continuous embedding $W \subset C(S; H)$ and the compact embedding $W \subset Y$, it follows the existence of subsequences

$$\{y_{m_k}\}_{k\geq 1} \subset \{y_m\}_{m\geq 1}, \quad \{d_{m_k}\}_{k\geq 1} \subset \{d_m\}_{m\geq 1}$$

and elements $y \in W$, $d \in X^*$, for which the next convergences take place:

$$\begin{aligned} y_{m_k} &\rightharpoonup y \text{ in } W, \quad d_{m_k} \rightharpoonup d \text{ in } X^*, \\ y_{m_k}(t) &\rightharpoonup y(t) \text{ in } H \text{ for each } t \in S, \\ y_{m_k}(t) &\to y(t) \text{ in } H \text{ for a.e. } t \in S, \text{ as } k \to \infty. \end{aligned} \tag{7.41}$$

From here, as $y_{m_k}(0) = \bar{0}$, $\forall k \geq 1$, we have $y(0) = \bar{0}$ (Fig. 7.10).

Step 3d.

Let us prove that

$$y' = f - d. \tag{7.42}$$

Let $\varphi \in D(S),\ n \in \mathbf{N}$ and $h \in H_n$. Then for all $k \geq 1$ such that $m_k \geq n$, we have

$$\left(\int_S \varphi(\tau)(y'_{m_k}(\tau) + d_{m_k}(\tau))d\tau, h \right) = \langle y'_{m_k} + d_{m_k}, \psi \rangle,$$

where $\psi(\tau) = h \cdot \varphi(\tau) \in X_n \subset X$. Let us remark that here we use the properties of Bochner integral (see [11], Theorem IV.1.8). Since $H_{m_k} \supset H_n$, for $m_k \geq n$, we get $\langle y'_{m_k} + d_{m_k}, \psi \rangle = \langle f_{m_k}, \psi \rangle$. Therefore, for all $k \geq 1$ such that $m_k \geq n$,

$$\langle f_{m_k}, \psi \rangle = \left(\int_S \varphi(\tau) f(\tau) d\tau, h \right).$$

Hence, for all $k \geq 1$ such that $m_k \geq n$,

$$\begin{aligned} \left(\int_S \varphi(\tau) y'_{m_k}(\tau) d\tau, h \right) &= \langle f - d_{m_k}, \psi \rangle \\ &\to \left(\int_S \varphi(\tau)(f(\tau) - d(\tau))d\tau, h \right) \text{ as } k \to \infty. \end{aligned} \tag{7.43}$$

The last convergence follows from the weak convergence d_{m_k} to d in X^*. From (7.41), we have

$$\left(\int_S \varphi(\tau) y'_{m_k}(\tau) d\tau, h \right) \to \big(y'(\varphi), h \big) \text{ as } k \to +\infty, \tag{7.44}$$

where

$$y'(\varphi) = -y(\varphi') = -\int_S y(\tau)\varphi'(\tau)d\tau,\ \forall \varphi \in \mathscr{D}(S).$$

Therefore, from (7.43) and (7.44), it follows that for all $\varphi \in \mathscr{D}(S),\ h \in \bigcup_{m \geq 1} H_m$,

$$\big(y'(\varphi), h\big) = \left(\int_S \varphi(\tau)(f(\tau) - d(\tau))d\tau, h \right).$$

Since $\bigcup_{m \geq 1} H_m$ is dense in V, we have that

$$y'(\varphi) = \int_S \varphi(\tau)(f(\tau) - d(\tau))d\tau, \forall \varphi \in \mathscr{D}(S).$$

Therefore, $y' = f - d \in X^*$.

Step 3e.

In order to prove that y is a solution of problem (7.29), it remains to show that y satisfies the inclusion $y' + A(y) \ni f$, and in virtue of (7.42), it is enough to prove that $d \in A(y)$.

From (7.41), it follows the existence of $\{\tau_l\}_{l \geq 1} \subset S$ such that $\tau_l \nearrow T$ as $l \to +\infty$ and

$$y_{m_k}(\tau_l) \to y(\tau_l) \text{ in } H, \forall l \geq 1, \text{ as } k \to +\infty. \tag{7.45}$$

Let us show that

$$\langle d, w\rangle \leq [A(y), w]_+, \tag{7.46}$$

for any $l \geq 1$ and $w \in X$ such that $w(t) = 0$ for a.e. $t \in [\tau_l, T]$.

Let us fix an arbitrary $\tau \in \{\tau_l\}_{l \geq 1}$. Let us set

$$X(\tau) = L_2(\tau, T; V), \quad X^*(\tau) = L_2(\tau, T; V^*),$$

$$\langle u, v\rangle_{X(\tau)} = \int_\tau^T \langle u(s), v(s)\rangle_V \, ds,$$

for $u \in X(\tau),\ v \in X^*(\tau)$, and

$$W(\tau) = \{u \in X(\tau) \mid u' \in X^*(\tau)\},$$

$$a_0 = y(\tau), \quad a_k = y_{m_k}(\tau), \quad k \geq 1.$$

From (7.45), it follows that

$$a_k \to a_0 \text{ in } H \text{ as } k \to +\infty. \tag{7.47}$$

For any $k \geq 1$, let $z_k \in W(\tau)$ be such that

$$\begin{cases} z_k' + J(z_k) = \bar{0}, \\ z_k(\tau) = a_k, \end{cases} \tag{7.48}$$

where $J : X(\tau) \to X^*(\tau)$ is the duality mapping (which is single-valued, as $X(\tau)$ is a Hilbert space), that is,

$$\langle J(u), u\rangle_{X(\tau)} = \|u\|^2_{X(\tau)} = \|J(u)\|^2_{X^*(\tau)}, \quad u \in X(\tau).$$

We remark that problem (7.48) has a solution $z_k \in W(\tau)$ because $J : X(\tau) \to X^*(\tau)$ is monotone, coercive, bounded, and demicontinuous (see [1, 2, 11, 18]). Let us also note that for any $k \geq 1$,

$$\|z_k(T)\|^2_H - \|a_k\|^2_H = 2\langle z'_k, z_k\rangle_{X(\tau)} = -2\|z_k\|^2_{X(\tau)}.$$

Hence,

$$\|z'_k\|_{X^*(\tau)} = \|z_k\|_{X(\tau)} \leq \frac{1}{\sqrt{2}}\|a_k\|_H \leq c_3, \ \forall k \geq 1.$$

Due to (7.47), similarly to [11, 18], z_k converges weakly in W, as $k \to +\infty$, to the unique solution $z_0 \in W$ of problem (7.48) with initial condition $z(\tau) = a_0$. Moreover,

$$z_k \to z_0 \text{ in } X(\tau) \text{ as } k \to +\infty, \tag{7.49}$$

because $\limsup\limits_{k\to+\infty} \|z_k\|^2_{X(\tau)} \leq \|z_0\|^2_{X(\tau)}$, $z_k \rightharpoonup z_0$ in $X(\tau)$, and $X(\tau)$ is a Hilbert space. For any $k \geq 1$, let us set

$$u_k(t) = \begin{cases} y_{m_k}(t), & \text{if } t \in [0, \tau], \\ z_k(t), & \text{elsewhere}, \end{cases}$$

$$g_k(t) = \begin{cases} d_{m_k}(t), & \text{if } t \in [0, \tau], \\ \hat{d}_k(t), & \text{elsewhere}, \end{cases}$$

where $\hat{d}_k \in A(u_k)$ be an arbitrary. As $\{u_k\}_{k\geq1}$ is bounded and $A : X \rightrightarrows X^*$ is bounded, we obtain that $\{\hat{d}_k\}_{k\geq1}$ is bounded in X^*. In virtue of (7.49), (7.41), and (7.45), we have

$$\begin{aligned}
\lim_{k\to+\infty}\langle g_k, u_k - u\rangle &= \lim_{k\to+\infty}\int_0^\tau \big(d_{m_k}(t), y_{m_k}(t) - y(t)\big)\,dt \\
&= \lim_{k\to+\infty}\int_0^\tau \big(f(t) - y'_{m_k}(t), y_{m_k}(t) - y(t)\big)\,dt \\
&= \lim_{k\to+\infty}\int_0^\tau \big(y'_{m_k}(t), y(t) - y_{m_k}(t)\big)\,dt
\end{aligned}$$

$$= \lim_{k\to+\infty} \frac{1}{2}\left(\|y_{m_k}(0)\|_H^2 - \|y_{m_k}(\tau)\|_H^2\right) + \lim_{k\to+\infty} \int_0^\tau \left(y'_{m_k}(t), y(t)\right) dt$$

$$= \frac{1}{2}\left(\|y(0)\|_H^2 - \|y(\tau)\|_H^2\right) + \int_0^\tau \left(y'(t), y(t)\right) dt = 0.$$

So,

$$\lim_{k\to+\infty} \langle g_k, u_k - u\rangle = 0. \tag{7.50}$$

Let us show that $g_k \in A(u_k)$, $\forall k \geq 1$. For any $w \in X$, let us set

$$w_\tau(t) = \begin{cases} w(t), & \text{if } t \in [0,\tau], \\ \bar{0}, & \text{elsewhere,} \end{cases}$$

$$w^\tau(t) = \begin{cases} \bar{0}, & \text{if } t \in [0,\tau], \\ w(t), & \text{elsewhere.} \end{cases}$$

Since A is an operator of the Volterra type, we obtain that

$$\langle g_k, w\rangle = \langle d_{m_k}, w_\tau\rangle + \langle \hat{d}_k, w^\tau\rangle \leq [A(y_{m_k}), w_\tau]_+ + \langle \hat{d}_k, w^\tau\rangle$$
$$= [A(u_k), w_\tau]_+ + \langle \hat{d}_k, w^\tau\rangle \leq [A(u_k), w_\tau]_+ + [A(u_k), w^\tau]_+.$$

Since $A(u_k) \in \mathscr{H}(X^*)$, similarly to the proof of (7.32), we obtain that

$$[A(u_k), w_\tau]_+ + [A(u_k), w^\tau]_+ = [A(u_k), w]_+.$$

As $w \in X$ be an arbitrary, we get $g_k \in A(u_k)$, for all $k \geq 1$. Since $\{u_k\}_{k\geq 1}$ is bounded in X, $\{g_k\}_{k\geq 1}$ is bounded in X^*. Thus, up to a subsequence $\{u_{k_j}, g_{k_j}\}_{j\geq 1} \subset \{u_k, g_k\}_{k\geq 1}$, for some $u \in W$, $g \in X^*$, the next convergence holds

$$u_{k_j} \rightharpoonup u \text{ in } W, \quad g_{k_j} \rightharpoonup g \text{ in } X^* \text{ as } j \to \infty. \tag{7.51}$$

We remark that

$$u(t) = y(t),\ g(t) = d(t) \text{ for a.e. } t \in [0,\tau]. \tag{7.52}$$

In virtue of (7.50), (7.51), as A satisfies the property S_k on W, we obtain that $g \in A(u)$. Hence, due to (7.52), as A is the Volterra type operator, for any $w \in X$ such that $w(t) = 0$ for a.e. $t \in [\tau, T]$ we get

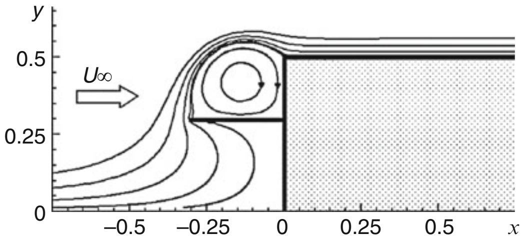

Fig. 7.11 The pattern of flow lines with forming of the standing vortex in front of the long body with the control plates on the windward side (see Sect. 6.4 and [41, Appendix A])

$$\langle d, w\rangle = \langle g, w\rangle \le [A(u), w]_+ = [A(y), w]_+.$$

As $\tau \in \{\tau_l\}_{l\ge 1}$ is an arbitrary, we obtain (7.46).

From (7.46), as the functional $w \to [A(y), w]_+$ is convex and lower semicontinuous on X (hence, it is continuous on X), we obtain that $\langle d, w\rangle \le [A(y), w]_+$, for any $w \in X$. So, $d \in A(y)$ (Fig. 7.11).

The theorem is proved. □

In a standard way (see [26]), by using the results of Theorem 7.2, we can obtain the following proposition.

Proposition 7.2. *Let $A : X \to C_v(X^*) \cap \mathscr{H}(X^*)$ be bounded map of the Volterra type which satisfies the property S_k on W. Moreover, let for some $\lambda_A \ge 0$ and $c > 0$,*

$$\frac{[A(y), y]_+ - c\|A(y)\|_+ + \lambda_A\|y\|_Y^2}{\|y\|_X} \to +\infty \tag{7.53}$$

as $\|y\|_X \to +\infty$. Then for any $a \in H$, $f \in X^*$ there exists at least one solution of problem (7.29), which can be obtained by the Faedo–Galerkin method.

Proof. Let us set $\varepsilon = \frac{\|a\|_H^2}{2c^2}$. We consider $w \in W$ such that

$$\begin{cases} w' + \varepsilon J(w) = \bar{0} \\ w(0) = a, \end{cases}$$

where $J : X \to X^*$ is the duality map. Hence, $\|w\|_X \le c$. We define $\hat{A} : X \to C_v(X^*) \cap \mathscr{H}(X^*)$ by the rule: $\hat{A}(z) = A(z + w)$, $z \in X$. Let us set $\hat{f} = f - w'^*$. If $z \in W$ is a solution of the problem

$$\begin{cases} z' + \hat{A}(z) \ni \hat{f} \\ z(0) = \bar{0}, \end{cases}$$

then $y = z + w$ is a solution of problem (7.29). It is clear that $\hat{A}$ is a bounded map of the Volterra type which satisfies the property S_k on W. So, due to Theorem 7.2, it is enough to prove the +-coercivity for the map $\hat{A} + \lambda_A I$. This property follows from the estimates:

$$
\begin{aligned}
[\hat{A}(z), z]_+ &\geq [A(z+w), z+w]_+ - [A(z+w), w]_+ \\
&\geq [A(z+w), z+w]_+ - c\|A(z+w)\|_+, \\
\|z\|_Y^2 &\geq \|z+w\|_Y^2 - c^2 - 2\|w\|_{X^*}\|z\|_X. \\
\|z\|_X &\geq \|z+w\|_X - c.
\end{aligned}
$$

The proposition is proved. □

Analyzing the proof of Theorem 7.2, we can obtain the following result.

Corollary 7.5. *Let $A : X \to C_v(X^*) \cap \mathscr{H}(X^*)$ be a bounded map of the Volterra type which satisfies the property S_k on W. We consider a sequence $\{a_n\}_{n\geq 0} \subset H$ such that $a_n \to a_0$ in H, as $n \to +\infty$. Let $y_n \in W$, $n \geq 1$, be solutions of problem (7.29) corresponding to the initial data a_n. If $y_n \rightharpoonup y_0$ in X, as $n \to +\infty$, then $y_0 \in W$ is solution of problem (7.29) with initial data a_0. Moreover, up to a subsequence, $y_n \rightharpoonup y_0$ in $W \subset C(S; H)$.*

7.2.4 Applications

Let us consider the bounded domain $\Omega \subset \mathbf{R}^n$ with smooth boundary $\partial\Omega$, $S = [0, T]$, $Q = \Omega \times (0; T)$, $\Gamma_T = \partial\Omega \times (0; T)$. For $a, b \in \mathbf{R}$ we set $[a, b] = \{\alpha a + (1-\alpha)b \mid \alpha \in [0, 1]\}$. Let $\overline{\theta}, \underline{\theta} : \mathbf{R} \to \mathbf{R}$ be real functions such that

$$-C(1+|s|) \leq \underline{\theta}(s) \leq \overline{\theta}(s) \leq C(1+|s|), \ \forall s \in \mathbf{R},$$

for some $C > 0$. We assume that $\overline{\theta}$ is upper semicontinuous and $\underline{\theta}$ is lower semicontinuous (Figs. 7.12–7.14).

Let $V = H_0^1(\Omega)$, $V^* = H^{-1}(\Omega)$ be its dual space, and $H = L_2(\Omega)$, $a \in H$, $f \in X^*$. We consider the problem:

$$
\begin{cases}
\frac{\partial y(x,t)}{\partial t} + [-\triangle y(x,t), \triangle y(x,t)] + \\
+[\underline{\theta}(y(x,t)), \overline{\theta}(y(x,t))] \ni f(x,t) \text{ in } Q, \\
y(x,0) = a(x), \ \text{in } \Omega, \\
y(x,t) = 0, \ \text{in } \Gamma_T.
\end{cases}
\tag{7.54}
$$

We define the maps $A_0 : X \to C_v(X^*) \cap \mathscr{H}(X^*)$, $A_1 : Y \to C_v(Y^*) \cap \mathscr{H}(Y^*)$ by

$$
\begin{aligned}
A_0(y) &= \{\triangle y \cdot p \mid p \in L_\infty(S), \ |p(t)| \leq 1 \text{ a.e. in } S\}, \\
A_1(y) &= \{d \in Y^* \mid d(x,t) \in [\underline{\theta}(y(x,t)), \overline{\theta}(y(x,t))] \text{ a.e. in } S\},
\end{aligned}
$$

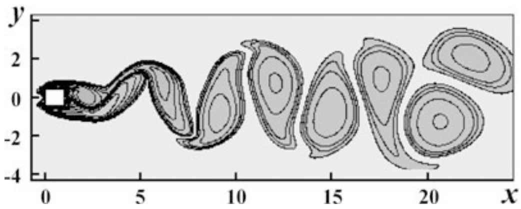

Fig. 7.12 Vorticity distribution in the wake of the square cylinder with control plates in the optimal regime ($l = 0.2,\ r = 0.16$) at Re $= 500$, $\tau = 25$ (see Sect. 6.4 and [41, Appendix A])

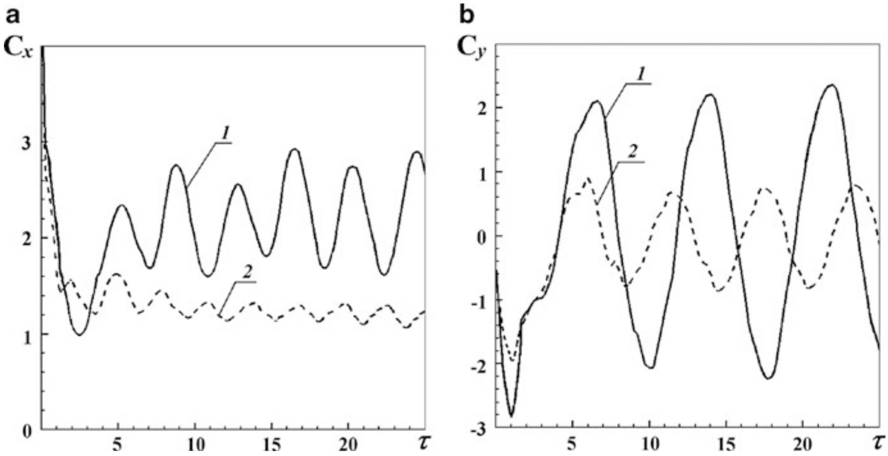

Fig. 7.13 Time dependencies of drag coefficient C_x – ***a*** and lifting force coefficient C_y – ***b*** for the square cylinder at Re $= 500$: ***1*** – with control, ***2*** – with optimal control ($l = 0.2,\ r = 0.16$) (see Sect. 6.4 and [41, Appendix A])

where $\triangle$ means the energetic extension in X of the Laplacian (see [11] for details) and $(\triangle y \cdot p)(x,t) = \triangle y(x,t) \cdot p(t)$ for a.e. $(x,t) \in Q$.

We note that there exists $C_1 > 0$ such that

$$\|A_0(y)\|_+ = \|y\|_X,\ [A_0(y), y]_+ = \|y\|_X^2, \tag{7.55}$$

$$\|A_1(y)\|_+ \le C_1(1 + \|y\|_Y),\ \forall y \in Y. \tag{7.56}$$

Also, the operator $A_1 : Y \to C_v(Y^*)$ is demiclosed.

We rewrite problem (7.54) in the following way (see [11] for details):

$$y' + A(y) \ni f,\ y(0) = a, \tag{7.57}$$

where $A : X \to C_v(X^*) \cap \mathscr{H}(X^*)$ is defined by $A(y) = A_0(y) + A_1(y)$, $y \in X$. A solution of problem (7.57) is called a generalized solution of (7.54).

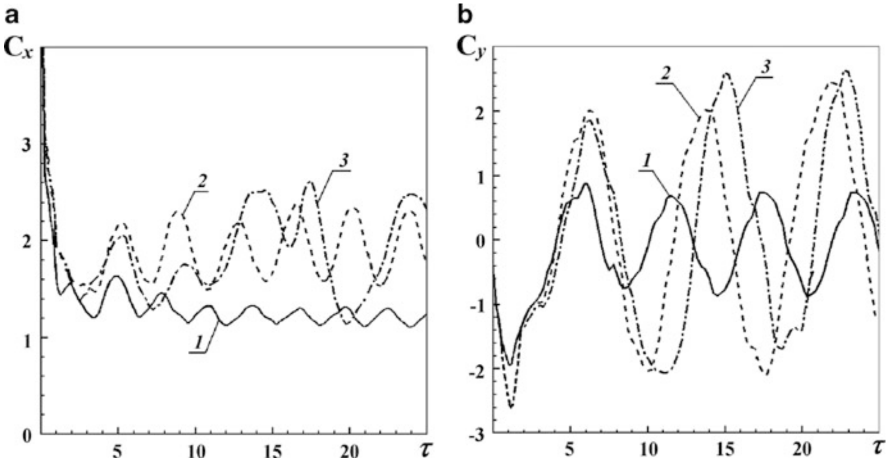

Fig. 7.14 Time dependencies of drag coefficient C_x – ***a*** and lifting force coefficient C_y – ***b*** for the square cylinder at different parameters of control plates: ***1*** – l=0.2, $r = 0.16$, ***2*** – $l = 0.2$, $r = 0.4$ ***3*** – $l = 0.2$, $r = 0.075$, $\mathrm{Re} = 500$ (see Sect. 6.4 and [41, Appendix A])

Due to Proposition 7.2, (7.55), (7.56), and the compact embedding $W \subset Y$, it is enough to check that A_0 satisfies the property S_k on W.

Indeed, let $y_n \rightharpoonup y$ in W, $d_n \rightharpoonup d$ in X^*, where $d_n = \triangle y_n \cdot p_n$, $p_n \in L_\infty(S)$, $|p_n(t)| \leq 1$ for a.e. $t \in S$. Then $y_n \to y$ in Y and up to a subsequence $p_n \to p$ weakly star in $L_\infty(S)$, where $|p(t)| \leq 1$ for a.e. $t \in S$. As $\|\triangle y_n \cdot p_n - \triangle y \cdot p_n\|_{L_2(S;H^{-2}(\Omega))} \leq \|y_n - y\|_Y \to 0$, we obtain $p_n \triangle y_n \to p \triangle y$ weakly in $L_2(S; H^{-2}(\Omega))$. Due to the continuous embedding $X^* \subset L_2(S; H^{-2}(\Omega))$, we obtain that $d = \triangle y \cdot p \in A_0(y)$.

We have:

Proposition 7.3. *Under the above conditions, problem (7.54) has at least one generalized solution $y \in W$.*

7.3 Functional-Topological Properties of the Resolving Operator for the Evolution Inclusion

For analysis and control for mathematical models of nonlinear geophysical processes and fields, in particular, piezoelectricity processes, unilateral contact problems in nonlinear elasticity and viscoelasticity, problems describing nonlinear frictional and adhesive effects, problems of delamination of plates, and loading and unloading problems in engineering structures (cf. Panagiotopoulos [34, 35], Naniewicz, Panagiotopoulos [32]), there is a necessity to develop corresponding noncoercive theory and high-precision algorithm for search of solutions (cf. [27]

and the references therein) as well as to study functional-topological properties of the resolving operator. We consider a series of results connected with properties of solutions of evolution inclusions with W_{λ_0}-quasimonotone maps and maps of $\bar{S}_k$ type.

7.3.1 The Setting of the Problem

Let $(V_i; H; V_i^*)$ be evolution triples such that for some counting set $\Phi \subset V = V_1 \cap V_2$

$$\Phi \text{ is dense in spaces } V,\ V_1,\ V_2 \text{ and in } H,$$
$$X = L_{r_1}(S;H) \cap L_{r_2}(S;H) \cap L_{p_1}(S;V_1) \cap L_{p_2}(S;V_2),$$
$$X^* = L_{q_1}(S;V_1^*) + L_{q_2}(S;V_2^*) + L_{r_2'}(S;H) + L_{r_1'}(S;H),$$
$$X_i = L_{r_i}(S;H) \cap L_{p_i}(S;V_i), \quad X_i^* = L_{q_i}(S;V_i^*) + L_{r_i'}(S;H),$$
$$W = \{y \in X \mid y' \in X^*\}, \quad W_i = \{y \in X_i \mid y' \in X^*\} \quad i = 1, 2$$

with corresponding norms as $p_0 := \max\{r_1, r_2\} < +\infty$ (see Corollary A.2), $S = [0, T]$. Note that $\langle\cdot,\cdot\rangle$ is the pairing on $X^* \times X$, that coincides with the inner product in $\mathcal{H} = L_2(S; H)$ on $\mathcal{H} \times X$. Let further Y be a reflexive or separable normalized space; Y^* be its dual space; U be a nonempty, convex, weakly star closed set in Y^*; and $A : X \times U \rightrightarrows X^*$ be a multivalued (in the general case) map. For fixed $f \in X^*$, $u \in U$, $a \in H$, the totality of solutions of such problem:

$$\begin{cases} y' + A(y,u) \ni f, \\ y(0) = a,\ y \in W \subset C(S;H) \end{cases} \tag{7.58}$$

we denote by $K(f, a, u)$.

7.3.2 Main Results

We present results connected with properties of the resolving operator for the differential-operator inclusion (Fig. 7.15).

Theorem 7.3. *Let $A : X \times U \to C_v(X^*)$ be a bounded λ_0-quasimonotone [3] on $W \times U$ map. Suppose that $\{f_m, a_m, u_m, y_m\}_{m\ge 1} \subset X^* \times H \times U \times W$:*

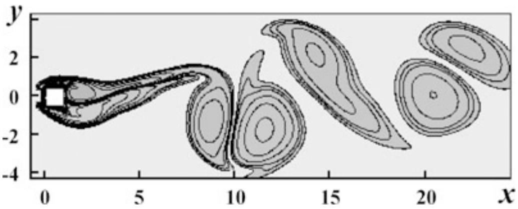

Fig. 7.15 Vorticity distribution in the wake of the square cylinder with control plates in nonoptimal regime ($l = 0.2$, $r = 0.075$) at Re $= 500$, $\tau = 25$ (see Sect. 6.4 and [41, Appendix A])

$$f_m \to f \text{ strongly in } X^* \text{ as } m \to +\infty, \tag{7.59}$$

$$a_m \to a \text{ strongly in } H \text{ as } m \to +\infty, \tag{7.60}$$

$$u_m \to u \text{ weakly star in } Y^* \text{ as } m \to +\infty, \tag{7.61}$$

$$y_m \to y \text{ weakly in } X \text{ as } m \to +\infty, \tag{7.62}$$

and

$$\forall m \geq 1 \quad y_m \in K(f_m, a_m, u_m). \tag{7.63}$$

Then

$$y \in K(f, a, u). \tag{7.64}$$

Proof. Let conditions of the theorem are fulfilled, $\{f_m, a_m, u_m, y_m\}_{m\geq 1} \subset X^* \times H \times U \times W$ be such sequences, that (7.59)–(7.63) hold true. Let us prove (7.64). From (7.59) to (7.62), it follows that

$$\exists R > 0 : \quad \forall m \geq 1 \quad \|f_m\|_{X^*} + \|a_m\|_H + \|y_m\|_X + \|u_m\|_{Y^*} \leq R. \tag{7.65}$$

From the inclusion from (7.63), it follows that $\forall m \geq 1\ \exists d_m \in A(y_m, u_m)$:

$$d_m = f_m - y_m' \in A(y_m, u_m). \tag{7.66}$$

The boundedness of $\{d_m\}_{m\geq 1}$ in X^* follows from the boundedness of A and from (7.65). Therefore,

$$\exists c_1 > 0 : \quad \forall m \geq 1 \quad \|d_m\|_{X^*} \leq c_1. \tag{7.67}$$

The boundedness of $\{y_m'\}_{m\geq 1}$ in X^* follows from (7.65)–(7.67). Hence,

$$\exists c_2 > 0 : \quad \forall m \geq 1 \quad \|y_m'\|_{X^*} \leq \|y_m\|_W \leq c_2. \tag{7.68}$$

Since the embedding $W \subset C(S; H)$ is continuous one (see Corollary A.1), in consequence of (7.68), we have that

$$\exists c_3 > 0 : \quad \forall m \geq 1, \ \forall t \in S \quad \|y_m(t)\|_H \leq c_3. \tag{7.69}$$

In view of estimates (7.65), (7.67)–(7.69), Banach–Alaoglu theorem, taking into account the reflexivity of X, there exist subsequences

$$\{y_{m_k}\}_{k\geq 1} \subset \{y_m\}_{m\geq 1}, \quad \{d_{m_k}\}_{k\geq 1} \subset \{d_m\}_{m\geq 1}$$

and an element $d \in X^*$, for which the convergences

$$y_{m_k} \rightharpoonup y \text{ in } W, \ d_{m_k} \rightharpoonup d \text{ in } X^*, \ \forall t \in S \ y_{m_k}(t) \rightharpoonup y(t) \text{ in } H \text{ as } k \to \infty \tag{7.70}$$

take place.

From here and from (7.60), (7.63), in particular, it follows that

$$y \in W \text{ and } y(0) = a. \tag{7.71}$$

Let us prove that

$$y' = f - d. \tag{7.72}$$

Let $\varphi \in D(S)$ and $h \in V$. Then $\forall k \geq 1$, we have

$$\begin{aligned}\left(\int_S \varphi(\tau)(y'_{m_k}(\tau) + d_{m_k}(\tau))d\tau, h\right) &= \int_S \Big(\varphi(\tau)(y'_{m_k}(\tau) + d_{m_k}(\tau)), h\Big)d\tau \\ &= \int_S \Big(y'_{m_k}(\tau) + d_{m_k}(\tau), \varphi(\tau)h\Big)d\tau \\ &= \langle y'_{m_k} + d_{m_k}, \psi\rangle,\end{aligned}$$

where $\psi(\tau) = h\cdot\varphi(\tau) \in X$. We remark that we use the property of Bochner integral here [11, Chap. IV]. Since for an arbitrary $k \geq 1$

$$\langle y'_{m_k} + d_{m_k}, \psi\rangle = \langle f_{m_k}, \psi\rangle,$$

we have $\forall k \geq 1$

$$\begin{aligned}\left(\int_S \varphi(\tau)y'_{m_k}(\tau)d\tau, h\right) = \left(\int_S \varphi(\tau)(f_{m_k}(\tau) - d_{m_k}(\tau))d\tau, h\right) = \int_S \Big((f_{m_k}(\tau) \\ - d_{m_k}(\tau)), \varphi(\tau)h\Big)d\tau = \langle f_{m_k} - d_{m_k}, \psi\rangle \to \langle f - d, \psi\rangle \\ = \left(\int_S \varphi(\tau)(f(\tau) - d(\tau))d\tau, h\right) \text{ as } k \to \infty. \end{aligned}\tag{7.73}$$

The last follows from the weak convergence of d_{m_k} to d in X^* and from (7.59).

From (7.70), we have

$$\left(\int_S \varphi(\tau)y'_{m_k}(\tau)d\tau, h\right) \to \left(\int_S \varphi(\tau)y'(\tau)d\tau, h\right) = \big(y'(\varphi), h\big) \text{ as } k \to +\infty, \tag{7.74}$$

where

$$\forall\varphi \in \mathscr{D}(S) \quad y'(\varphi) = -y(\varphi') = -\int_S y(\tau)\varphi'(\tau)d\tau.$$

Hence, from (7.73) and (7.74), it follows that

$$\forall\varphi \in \mathscr{D}(S)\ \forall h \in V \quad \big(y'(\varphi), h\big) = \left(\int_S \varphi(\tau)(f(\tau) - d(\tau))d\tau, h\right),$$

or

$$\forall\varphi \in \mathscr{D}(S) \quad y'(\varphi) = \int_S \varphi(\tau)(f(\tau) - d(\tau))d\tau.$$

Therefore, $y' = f - d \in X^*$.

In order to prove (7.64), it remains to show that y satisfies the inclusion $y' + A(y, u) \ni f$. In view of the identity (7.72), it sufficiently prove that $d \in A(y, u)$.

Firstly, we ensure that

$$\varlimsup_{k\to\infty} \langle d_{m_k}, y_{m_k} - y\rangle \le 0. \tag{7.75}$$

Indeed, in view of (7.72), $\forall k \ge 1$, we have

$$\begin{aligned}\langle d_{m_k}, y_{m_k} - y\rangle &= \langle f_{m_k}, y_{m_k}\rangle - \langle y'_{m_k}, y_{m_k}\rangle - \langle d_{m_k}, y\rangle \\ &= \langle f_{m_k}, y_{m_k}\rangle - \langle d_{m_k}, y\rangle + \frac{1}{2}\big(\|y_{m_k}(0)\|_H^2 - \|y_{m_k}(T)\|_H^2\big).\end{aligned} \tag{7.76}$$

Further, for the left and right parts of the equality (7.76), we pass the upper limit as $k \to \infty$. From here, we have

$$\begin{aligned}\varlimsup_{k\to\infty} \langle d_{m_k}, y_{m_k} - y\rangle &\le \varlimsup_{k\to\infty} \langle f_{m_k}, y_{m_k}\rangle + \varlimsup_{k\to\infty} \langle d_{m_k}, -y\rangle \\ &+ \varlimsup_{k\to\infty} \frac{1}{2}\big(\|y_{m_k}(0)\|_H^2 - \|y_{m_k}(T)\|_H^2\big) \le \langle f, y\rangle_X - \langle d, y\rangle \\ &+ \frac{1}{2}\big(\|y(0)\|_H^2 - \|y(T)\|_H^2\big) = \langle f - d, y\rangle - \langle y', y\rangle = 0.\end{aligned}$$

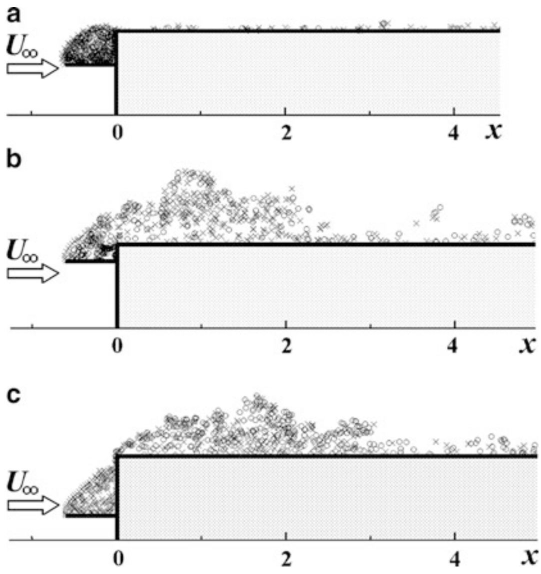

Fig. 7.16 Discrete-vortex model of the flow in front of the body of non-streamline form with control plates on the windward side: optimal (**a**) and nonoptimal (**b**, **c**) plate parameters (see Sect. 6.4 and [41, Appendix A])

The last holds true in consequence of (7.72), [11, Chap. I], (7.59) and (7.70). The inequality (7.75) is checked.

From conditions (7.70), (7.75), and λ_0-quasimonotony of A on $W \times U$, it follows that there exist such $\{d_l\} \subset \{d_{m_k}\}_{k\geq 1}$, $\{y_l\} \subset \{y_{m_k}\}_{k\geq 1}$, that

$$\forall \omega \in X \qquad \underline{\lim_{l\to+\infty}} \langle d_l, y_l - \omega\rangle \ \geq\ [A(y,u), y-\omega]_- \,, \tag{7.77}$$

in particular, taking into account (7.75), we obtain that

$$\lim_{l\to+\infty} \langle d_l, y_l - y\rangle = 0. \tag{7.78}$$

From here, from (7.77) and (7.70), we have

$$\forall \omega \in X \quad [A(y,u), \omega - y]_+ \geq \langle d, \omega - y\rangle$$

that is equivalent to $d \in A(y,u)$. Therefore, $y \in K(f,a,u)$.

The theorem is proved (Fig. 7.16). □

Theorem 7.4. *Let the space V_i, $i = 1,2$ be uniformly convex one* (*cf.* [11, p. 21]), *the embedding $V \subset H$ is compact one, $A : X \times U \to C_v(X^*) \cap \mathscr{H}(X^*)$ be a bounded map that satisfies Property $\bar{S}_k$ on $W \times U$ [3]. Moreover, let for each $u \in U$ the $A(\cdot,u)$ be the Volterra type operator. Suppose that $\{f_m, a_m, u_m, y_m\}_{m\geq 1} \subset X^* \times H \times U \times W$:*

$$f_m \to f \textit{ strongly in } X^* \textit{ as } m \to +\infty, \tag{7.79}$$

$$a_m \to a \textit{ strongly in } H \textit{ as } m \to +\infty, \tag{7.80}$$

$$u_m \to u \textit{ weakly star in } Y^* \textit{ as } m \to +\infty, \tag{7.81}$$

$$y_m \to y \textit{ weakly in } X \textit{ as } m \to +\infty, \tag{7.82}$$

and

$$\forall m \geq 1 \quad y_m \in K(f_m, a_m, u_m). \tag{7.83}$$

Then

$$y \in K(f, a, u). \tag{7.84}$$

Proof. Let conditions of the theorem are fulfilled, $\{f_m, a_m, u_m, y_m\}_{m \geq 1} \subset X^* \times H \times U \times W$ be such sequences that (7.79)–(7.83) hold true. Let us prove (7.84). From (7.79) to (7.82), it follows that

$$\exists R > 0: \quad \forall m \geq 1 \quad \|f_m\|_{X^*} + \|a_m\|_H + \|y_m\|_X + \|u_m\|_{Y^*} \leq R. \tag{7.85}$$

From the inclusion from (7.58), it follows that $\forall m \geq 1\ \exists d_m \in A(y_m, u_m):$

$$d_m = f_m - y'_m \in A(y_m, u_m). \tag{7.86}$$

The boundedness of $\{d_m\}_{m \geq 1}$ in X^* follows from the boundedness of A and from (7.85). Hence,

$$\exists c_1 > 0: \quad \forall m \geq 1 \quad \|d_m\|_{X^*} \leq c_1. \tag{7.87}$$

The boundedness of $\{y'_m\}_{m \geq 1}$ in X^* follows from (7.85), (7.86). Hence,

$$\exists c_2 > 0: \quad \forall m \geq 1 \quad \|y'_m\|_{X^*} \leq \|y_m\|_W \leq c_2. \tag{7.88}$$

In view of the continuity of the embedding $W \subset C(S; H)$ (see Corollary A.1) we obtain that

$$\exists c_3 > 0: \quad \forall m \geq 1,\ \forall t \in S \quad \|y_m(t)\|_H \leq c_3. \tag{7.89}$$

From (7.82), (7.85), and (7.87)–(7.89), in view of Banach–Alaoglu theorem, taking into account the compactness of the embedding $W \subset \mathscr{H}$ (see [44], Corollary A.1), it follows the existence of subsequences

$$\{y_{m_k}\}_{k \geq 1} \subset \{y_m\}_{m \geq 1}, \quad \{d_{m_k}\}_{k \geq 1} \subset \{d_m\}_{m \geq 1}$$

and an element $d \in X^*$, for which the convergences

$$\begin{aligned} &y_{m_k} \rightharpoonup y \text{ in } W, \quad d_{m_k} \rightharpoonup d \text{ in } X^*, \\ &y_{m_k}(t) \rightharpoonup y(t) \text{ in } H \text{ for each } t \in S, \\ &y_{m_k}(t) \to y(t) \text{ in } H \text{ for a.e. } t \in S, \quad \text{as } k \to \infty, \end{aligned} \tag{7.90}$$

take place, in particular, $y \in W$. From here, since $\forall k \geq 1\ y_{m_k}(0) = a_{m_k}$, we have $y(0) = a$.

Let us prove that

$$y' = f - d. \tag{7.91}$$

Let $\varphi \in D(S)$ and $h \in V$. Then $\forall k \geq 1$, we have

$$\left(\int_S \varphi(\tau)(y'_{m_k}(\tau) + d_{m_k}(\tau))d\tau, h \right) = \langle y'_{m_k} + d_{m_k}, \psi \rangle,$$

where $\psi(\tau) = h \cdot \varphi(\tau) \in X$. We remark that we use the property of Bochner integral here [11, Chap. IV]. Since for $k \geq 1$ $\langle y'_{m_k} + d_{m_k}, \psi \rangle = \langle f_{m_k}, \psi \rangle$, we obtain

$$\langle f_{m_k}, \psi \rangle = \left(\int_S \varphi(\tau) f_{m_k}(\tau) d\tau, h \right).$$

and

$$\left(\int_S \varphi(\tau) y'_{m_k}(\tau) d\tau, h \right) = \langle f_{m_k} - d_{m_k}, \psi \rangle$$

$$\to \left(\int_S \varphi(\tau)(f(\tau) - d(\tau))d\tau, h \right) \text{ as } k \to \infty. \tag{7.92}$$

The last follows from the weak convergence of d_{m_k} to d in X^* and from (7.79).

From (7.90), we have

$$\left(\int_S \varphi(\tau) y'_{m_k}(\tau) d\tau, h \right) \to \big(y'(\varphi), h\big) \text{ as } k \to +\infty, \tag{7.93}$$

where

$$\forall \varphi \in \mathscr{D}(S) \quad y'(\varphi) = -y(\varphi') = -\int_S y(\tau)\varphi'(\tau)d\tau.$$

Therefore, from (7.92) and (7.93), it follows that

$$\forall \varphi \in \mathscr{D}(S)\ \forall h \in V \quad \big(y'(\varphi), h\big) = \left(\int_S \varphi(\tau)(f(\tau) - d(\tau))d\tau, h \right),$$

that is,

$$\forall \varphi \in \mathscr{D}(S) \quad y'(\varphi) = \int_S \varphi(\tau)(f(\tau) - d(\tau))d\tau.$$

Hence, $y' = f - d \in X^*$.

In order to prove (7.84), it remains to show that y satisfies the inclusion $y' + A(y,u) \ni f$. In consequence of the identity (7.91), it sufficiently show that $d \in A(y,u)$.

From (7.90), it follows the existence of $\{\tau_l\}_{l\geq 1} \subset S$ such that $\tau_l \nearrow T$ as $l \to +\infty$ and

$$\forall l \geq 1 \quad y_{m_k}(\tau_l) \to y(\tau_l) \text{ in } H \text{ as } k \to +\infty. \tag{7.94}$$

Show that for an arbitrary $l \geq 1$,

$$\langle d, w\rangle \leq [A(y,u), w]_+ \quad \forall w \in X : \ w(t) = 0 \text{ for a.e. } t \in [\tau_l, T]. \tag{7.95}$$

Let us fix an arbitrary $\tau \in \{\tau_l\}_{l\geq 1}$. For $i = 1, 2$, we set

$$X(\tau) = L_{r_1}(\tau, T; H) \cap L_{p_1}(\tau, T; V_1) \cap L_{r_2}(\tau, T; H) \cap L_{p_2}(\tau, T; V_2),$$
$$X^*(\tau) = L_{r_1'}(\tau, T; H) + L_{q_1}(\tau, T; V_1^*) + L_{r_2'}(\tau, T; H) + L_{q_2}(\tau, T; V_2^*),$$
$$W(\tau) = \{y \in X(\tau) \mid y' \in X^*(\tau)\},$$
$$\langle u, v\rangle_{X(\tau)} = \int_\tau^T (u(s), v(s))ds, \quad u \in X^*(\tau), v \in X(\tau),$$
$$b_0 = y(\tau), \quad b_k = y_{m_k}(\tau), \quad k \geq 1,$$

where y' is the derivative of $y \in X(\tau)$ in the sense of $\mathscr{D}(\tau, T; V^*)$ (Fig. 7.17).

From (7.94), it follows that

$$b_k \to b_0 \text{ in } H \text{ as } k \to +\infty. \tag{7.96}$$

For an arbitrary $k \geq 1$, we show that there exists such $z_k \in W(\tau)$ that

$$\begin{cases} z_k' + J(z_k) \ni \bar{0}, \\ z_k(\tau) = b_k, \end{cases} \tag{7.97}$$

where z_k' is the derivative of $z_k \in X(\tau)$ in the sense of $\mathscr{D}(\tau, T; V^*)$,

$$J = \partial\left(\frac{\|\cdot\|^2_{X(\tau)}}{2}\right) : X(\tau) \to C_v(X^*(\tau))$$

is dual (multivalued in the general case) map, that is (cf. [1, 18]), $\forall u \in X(\tau)$

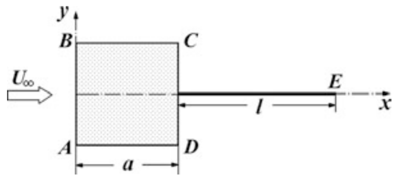

Fig. 7.17 Diagram of the flow past the square prism with the separating plate (see Sect. 6.4 and [41, Appendix A])

$$J(u) = \left\{ p \in X^*(\tau) \mid \langle p, u\rangle_{X(\tau)} = \|u\|^2_{X(\tau)} = \|p\|^2_{X^*(\tau)} \right\}. \tag{7.98}$$

The solvability of problem (7.97) in the class $W(\tau)$ follows from such suggestions. From [19, Theorem 2, Theorem 3] and (7.98), it follows that such conditions hold true:

(i_1) $J : X(\tau) \to C_v(X^*(\tau))$ is a monotone map.
(i_2) $J : X(\tau) \to C_v(X^*(\tau))$ is an upper hemicontinuous map.
(i_3) $J : X(\tau) \to C_v(X^*(\tau))$ is a bounded map.
From (7.98), we also have that

$$\forall y \in X(\tau) \quad [J(y), y]_- - \|J(y)\|_+ = \|y\|^2_{X(\tau)} - \|y\|_{X(\tau)}.$$

Hence, the condition
(i_4) $J : X(\tau) \to C_v(X^*(\tau))$ satisfies such coercivity condition:

$$\frac{[J(y), y]_- - \|J(y)\|_+}{\|y\|_{X(\tau)}} \to +\infty \quad \text{as} \quad \|y\|_{X(\tau)} \to \infty$$

holds true.

So, from conditions i_1–i_4, problem (7.97) has at least one solution in the class $W(\tau)$ (cf. [49]). Note that in view of (7.98) and (A.11), for any $k \geq 1$,

$$\|z_k(T)\|^2_H - \|b_k\|^2_H = 2\langle z_k', z_k\rangle_{X(\tau)},$$

$$\langle z_k', z_k\rangle_{X(\tau)} + \|z_k\|^2_{X(\tau)} = 0.$$

Therefore,

$$\forall k \geq 1 \quad \|z_k'\|_{X^*(\tau)} = \|z_k\|_{X(\tau)} \leq \frac{1}{\sqrt{2}}\|b_k\|_H \leq c_3. \tag{7.99}$$

Moreover, in consequence of the continuity of the embedding $W(\tau) \subset C(\tau, T; H)$ (see Theorem A.6), we obtain that $\exists c_4 > 0$ such that

$$\forall k \geq 1,\ \forall s \in [\tau, T] \quad \|z_k(s)\|_H \leq c_4. \tag{7.100}$$

Show that in view of (7.96), as $k \to +\infty$, up to a subsequence, z_k weakly converges in $W(\tau)$ to some solution $z \in W(\tau)$ of problem (7.97) with initial condition $z(\tau) = b_0$. In view of estimates (7.99) and Banach–Alaoglu theorem (Theorem A.2), taking into account the reflexivity $X(\tau)$, it follows the existence of a subsequence $\{z_{k_l}\}_{l\geq 1} \subset \{z_k\}_{k\geq 1}$ and an element $z \in W(\tau)$, for which the convergence

$$z_{k_l} \rightharpoonup z \text{ in } W(\tau) \subset C(\tau, T; H) \text{ as } l \to \infty \tag{7.101}$$

takes place.

From here and from (7.96), (7.97), it, in particular, follows that $z(\tau) = b_0$.

We ensure that

$$\overline{\lim_{l\to\infty}} \langle -z'_{k_l}, z_{k_l} - z\rangle_{X(\tau)} \leq 0. \tag{7.102}$$

Indeed, in view of (7.97), $\forall l \geq 1$, we have

$$\langle -z'_{k_l}, z_{k_l} - z\rangle_{X(\tau)} = \langle z'_{k_l}, z\rangle_{X(\tau)} + \frac{1}{2}\left(\|z_{k_l}(\tau)\|_H^2 - \|z_{k_l}(T)\|_H^2\right). \tag{7.103}$$

The last we obtain in view of (A.11). Further, for the left and right parts of the equality (7.103), we pass to the upper limit as $l \to +\infty$. From here, we have

$$\begin{aligned}\overline{\lim_{l\to\infty}} \langle -z'_{k_l}, z_{k_l} - z\rangle_{X(\tau)} &\leq \overline{\lim_{l\to\infty}} \langle z'_{k_l}, z\rangle_{X(\tau)} + \overline{\lim_{l\to\infty}} \frac{1}{2}\left(\|z_{k_l}(\tau)\|_H^2 - \|z_{k_l}(T)\|_H^2\right)\\ &\leq \langle z', z\rangle_{X(\tau)} + \frac{1}{2}\left(\|z(\tau)\|_H^2 - \|z(T)\|_H^2\right)\\ &= \langle z' - z', z\rangle_{X(\tau)} = 0.\end{aligned}$$

The last is fulfilled in view of (7.101), the formula (A.11), and the inequality

$$\overline{\lim_{l\to\infty}} (-\|z_{k_l}(T)\|_H^2) \leq -\|z(T)\|_H^2$$

(cf. [11, Chap. I]). The inequality (7.102) is proved.

From conditions (7.101), (7.97), (7.102), and λ_0-pseudomonotony J on $W(\tau)$ (see i_1–i_3 and [44]), it follows that there exists such subsequence $\{z_j\}_j \subset \{z_{k_l}\}_{l\geq 1}$ that

$$\forall \omega \in X(\tau) \quad \underline{\lim_{j\to\infty}} \langle -z'_j, z_j - \omega\rangle_{X(\tau)} \geq [J(z), z - \omega]_- . \tag{7.104}$$

In particular, taking into account (7.102), we obtain that

$$\lim_{j\to\infty} \langle -z'_j, z_j - z\rangle_{X(\tau)} = 0.$$

From here, from (7.104), and from the convergence (7.101), we have

$$\forall \omega \in X(\tau) \quad \langle -z', z - \omega\rangle_{X(\tau)} \geq [J(z), z - \omega]_-$$

that is equivalent to $-z' \in J(z)$. Hence, $z \in W(\tau)$ is the solution of (7.97) with initial data $z(\tau) = b_0$.

Show that

$$z_{k_l} \to z \text{ in } X(\tau) \text{ as } l \to +\infty. \tag{7.105}$$

Firstly, we remark that

$$X(\tau) = Z_1 \cap Z_2 \cap Z_3 \cap Z_4,$$

where

$$Z_i = L_{r_i}(\tau, T; H), \quad Z_{i+2} = L_{p_i}(\tau, T; V_i), \quad i = 1, 2.$$

Since the space H is Hilbert one, it is uniformly convex one. The space $Z_i, i = 1, 4$ is also uniformly convex one. Suppose that (7.105) is not fulfilled. Then there exists such $i_0 = 1, 4$ that

$$z_{k_l} \not\to z \text{ in } Z_{i_0} \text{ as } l \to +\infty.$$

It means that

$$\|z\|_{Z_{i_0}} < \varliminf_{l \to +\infty} \|z_{k_l}\|_{Z_{i_0}}. \tag{7.106}$$

On the other hand, in consequence of (7.97)–(7.99) and (7.102), we have that

$$\varlimsup_{l \to +\infty} \|z_{k_l}\|^2_{X(\tau)} = \varlimsup_{l \to +\infty} \langle -z'_{k_l}, z_{k_l} \rangle_{X(\tau)} \le \langle -z', z \rangle_{X(\tau)} = \|z\|^2_{X(\tau)}.$$

Therefore,

$$\varlimsup_{l \to +\infty} \|z_{k_l}\|_{X(\tau)} \le \|z\|_{X(\tau)}.$$

From here and from (7.106),

$$\begin{aligned} \|z\|_{X(\tau)} &= \sum_{i=1}^{4} \|z\|_{Z_i} < \sum_{i=1}^{4} \varliminf_{l \to +\infty} \|z_{k_l}\|_{Z_i} \le \varliminf_{l \to +\infty} \sum_{i=1}^{4} \|z_{k_l}\|_{Z_i} \\ &= \varliminf_{l \to +\infty} \|z_{k_l}\|_{X(\tau)} \le \varlimsup_{l \to +\infty} \|z_{k_l}\|_{X(\tau)} \le \|z\|_{X(\tau)}. \end{aligned}$$

We have the contradiction. Hence, (7.105) is fulfilled.

For an arbitrary $l \ge 1$, we set

$$v_l(t) = \begin{cases} y_{m_{k_l}}(t), \text{ if } t \in [0, \tau], \\ z_{k_l}(t), \quad \text{else}, \end{cases} \qquad g_l(t) = \begin{cases} d_{m_{k_l}}(t), \text{ if } t \in [0, \tau], \\ \hat{d}_l(t), \quad \text{else}, \end{cases}$$

$$v(t) = \begin{cases} y(t), \text{ if } t \in [0, \tau], \\ z(t), \text{ else}, \end{cases}$$

where $\hat{d}_l \in A(v_l, u_{m_{k_l}})$ is an arbitrary one. Since $\{v_l, u_{m_{k_l}}\}_{l \ge 1}$ is bounded sequence in $X \times Y^*$ (cf. (7.90), (7.105), (7.85)), and the map $A : X \times U \overset{\to}{\to} X^*$ is bounded one, we have $\{\hat{d}_l\}_{l\geq 1}$ is bounded sequence in X^*. In consequence of (7.105), (7.90), (7.94)

$$\begin{aligned}
\lim_{l\to+\infty}\langle g_l, v_l - v\rangle_X &= \lim_{l\to+\infty}\int_0^\tau \Big(d_{m_{k_l}}(t), y_{m_{k_l}}(t) - y(t)\Big)\,dt \\
&= \lim_{l\to+\infty}\int_0^\tau \Big(f(t) - y'_{m_{k_l}}(t), y_{m_{k_l}}(t) - y(t)\Big)\,dt \\
&= \lim_{l\to+\infty}\int_0^\tau \Big(y'_{m_{k_l}}(t), y(t) - y_{m_{k_l}}(t)\Big)\,dt \\
&= \lim_{l\to+\infty}\frac{1}{2}\Big(\|y_{m_{k_l}}(0)\|_H^2 - \|y_{m_{k_l}}(\tau)\|_H^2\Big) \\
&\quad + \lim_{l\to+\infty}\int_0^\tau \Big(y'_{m_{k_l}}(t), y(t)\Big)\,dt \\
&= \frac{1}{2}\big(\|y(0)\|_H^2 - \|y(\tau)\|_H^2\big) + \int_0^\tau \big(y'(t), y(t)\big)\,dt = 0.
\end{aligned}$$

Therefore,

$$\lim_{l\to+\infty}\langle g_l, v_l - v\rangle = 0. \tag{7.107}$$

Show that $g_l \in A(v_l, u_{m_{k_l}})$ $\forall l \geq 1$. For an arbitrary $w \in X$, we set

$$w_\tau(t) = \begin{cases} w(t), & \text{if } t \in [0,\tau], \\ \bar{0}, & \text{else,} \end{cases} \qquad w^\tau(t) = \begin{cases} \bar{0}, & \text{if } t \in [0,\tau], \\ w(t), & \text{else.} \end{cases}$$

In consequence of $A(\cdot, u_{m_{k_l}})$ is the Volterra type operator, we obtain that

$$\begin{aligned}
\langle g_l, w\rangle &= \langle d_{m_{k_l}}, w_\tau\rangle + \langle \hat{d}_l, w^\tau\rangle \leq [A(y_{m_{k_l}}, u_{m_{k_l}}), w_\tau]_+ + \langle \hat{d}_l, w^\tau\rangle \\
&= [A(v_l, u_{m_{k_l}}), w_\tau]_+ + \langle \hat{d}_l, w^\tau\rangle \\
&\leq [A(v_l, u_{m_{k_l}}), w_\tau]_+ + [A(v_l, u_{m_{k_l}}), w^\tau]_+.
\end{aligned}$$

In consequence of $A(v_l, u_{m_{k_l}}) \in \mathscr{H}(X^*)$, we obtain that

$$[A(v_l, u_{m_{k_l}}), w_\tau]_+ + [A(v_l, u_{m_{k_l}}), w^\tau]_+ = [A(v_l, u_{m_{k_l}}), w]_+.$$

Since $w \in X$ is an arbitrary one, we have $g_l \in A(v_l, u_{m_{k_l}})$ $\forall l \geq 1$. In consequence of $\{v_l\}_{l\geq 1}$ is bounded one in W and $\{u_{m_{k_l}}\}_{l\geq 1}$ is bounded one in Y^*, we have that $\{g_l\}_{l\geq 1}$ is bounded sequence in X^*. Therefore, up to a subsequence $\{v_{l_j}, g_{l_j}\}_{j\geq 1} \subset \{v_l, g_l\}_{l\geq 1}$, for some $v \in W$, $g \in X^*$ such convergences (see (7.90), (7.99), (7.100))

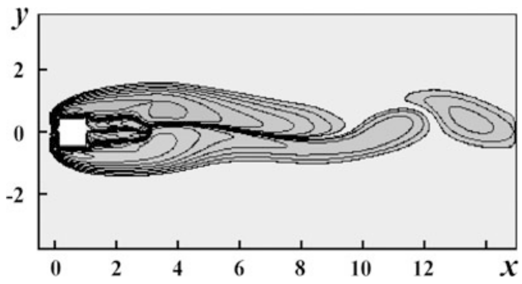

Fig. 7.18 Pattern of vorticity isolines at the flow past the square prism with the separating plate (see Sect. 6.4 and [41, Appendix A])

$$v_{l_j} \rightharpoonup v \text{ in } W, \quad g_{l_j} \rightharpoonup g \text{ in } X^* \text{ as } j \to \infty \tag{7.108}$$

take place (Fig. 7.18).

We remark that

$$v(t) = y(t), \; g(t) = d(t) \text{ for a.e. } t \in [0, \tau]. \tag{7.109}$$

In consequence of (7.81), (7.107), and (7.108), since A satisfies Property $\bar{S}_k$ on $W \times U$, we obtain that $g \in A(v, u)$. Hence, in view of (7.109), since $A(\cdot, u)$ is the Volterra type operator, for an arbitrary $w \in X$ such that $w(t) = 0$ for a.e. $t \in [\tau, T]$, we have

$$\langle d, w \rangle = \langle g, w \rangle \leq [A(v, u), w]_+ = [A(y, v), w]_+.$$

Since $\tau \in \{\tau_l\}_{l \geq 1}$ is an arbitrary one, we obtain (7.95).

From (7.95), in view of that the functional $w \to [A(y, u), w]_+$ is convex and lower semicontinuous on X (and, therefore, it is continuous on X [2]), we obtain that for an arbitrary $w \in X$ $\langle d, w \rangle \leq [A(y, u), w]_+$. Hence, $d \in A(y, u)$.

The theorem is proved. □

Remark 7.1. Statements of represented in this subsection theorems are fulfilled, in particular, in that case, when we consider strong topology instead weakly star topology in the space Y^*.

7.4 Auxiliary Properties of Solutions for the Nonautonomous First-Order Evolution Inclusions with Uniformly Coercive Mappings, Long-Time Behavior, and Pullback Attractors

In this section, we consider Nemytskii operator properties for classes of pointwise pseudomonotone multivalued maps, considered in [25] (see paper and references therein). We obtain these properties, analyzing theorem proofs from [25] and Sect. 2.2. At that, we consider weaker properties for operators connected with measurability and obtain stronger results that we use in further sections.

For evolution triple (V, H, V^*),[1] $p > 1$, multivalued (in the general case) map $A : S \times V \overset{\rightarrow}{\rightarrow} V^*$, $S = [\tau, T, +\infty]$ and exciting force $f \in V^*$ we consider a problem of investigation of dynamics for all weak solutions defined for $t \geq 0$ of nonlinear autonomous differential-operator inclusion

$$y'(t) + A(t, y(t)) \ni f, \tag{7.110}$$

as $t \to +\infty$ in the phase space H. Parameters of this problem satisfy the next properties: We suppose

$(A1)$ For a.e. $t \in S$ $v \to A(t, v)$ is a pseudomonotone map such that:

- (a) $A(t, u) \in C_v(V^*)$ for a.e. $t \in S$ and $\forall u \in V$, i.e., the set $A(t, u)$ is a nonempty, closed, and convex one for all $u \in V$.
- (b) If $u_j \to u$ weakly in V and $d_j \in A(t, u_j)$ is such that

$$\overline{\lim_{j\to+\infty}} \langle d_j, u_j - u\rangle_V \leq 0,$$

then

$$\underline{\lim_{j\to+\infty}} \langle d_j, u_j - \omega\rangle_V \geq [A(t, u), u - \omega]_- \ \forall \omega \in V.$$

$(A2)$ $\exists c_1 > 0$:

$$\|A(t, u)\|_+ \leq c_1(1 + \|u\|_V^{p-1}) \ \forall u \in V, \text{ for a.e. } t \in S.$$

$(A3)$ $\exists c_2, c_3 > 0$:

$$[A(t, u), u]_- \geq c_2\|u\|_V^p - c_3 \ \forall u \in V, \text{ for a.e. } t \in S.$$

$(A4)$ $A : S \times V \to C_v(V^*)$ is measurable [25].

As a *weak solution* of evolution inclusion (7.110) on the interval $[\tau, T]$, we consider an element u of the space $L_p(\tau, T; V)$ such that for some $d \in L_q(\tau, T; V^*)$

$$d(t) \in A(t, y(t)) \quad \text{for almost each (a.e.) } t \in (\tau, T), \tag{7.111}$$

$$-\int_\tau^T (\xi'(t), u(t))dt + \int_\tau^T \langle d(t), \xi(t)\rangle_V dt = \int_\tau^T (f, \xi(t))dt \quad \forall \xi \in C_0^\infty([\tau, T]; V), \tag{7.112}$$

where $q > 1$: $\frac{1}{p} + \frac{1}{q} = 1$.

We consider a reflexive separable Banach space V_σ such that $V_\sigma \subset V$ with dense and continuous embedding. Therefore, we have the chain of continuous and dense

[1] That is, V is a real reflexive separable Banach space embedded into a real Hilbert space H continuously and densely, H is identified with its conjugated space H^* and V^* is a dual space to V. So, we have such chain of continuous and dense embeddings: $V \subset H \equiv H^* \subset V^*$ (see, e.g., [49]).

embeddings:

$$V_\sigma \subset V \subset H \equiv H^* \subset V^* \subset V_\sigma^*,$$

where V_σ^* is dual space to V_σ. Let us set: $S = [\tau, T], -\infty < \tau < T < +\infty, q > 1 : \frac{1}{p} + \frac{1}{q} = 1,$

$$X = L_p(S; V),\ X^* = L_q(S; V^*),\ X_\sigma = L_p(S; V_\sigma),\ X_\sigma^* = L_q(S; V_\sigma^*),$$
$$W = \{y \in X \mid y' \in X^*\},\ W_\sigma = \{y \in X \mid y' \in X_\sigma^*\}.$$

Analyzing the proof of lemmas from Sect. 2.2, we get:

Lemma 7.8. *Under above conditions for any $y \in X$,*

$$\hat{A}(y) = \{g \in X^* |\ g(t) \in A(t, y(t)) \text{ for a.e. } t \in S\} \neq \emptyset.$$

Moreover, $\hat{A}$ is pseudomonotone on W_σ,

$$\exists C_1 > 0 : \quad \|\hat{A}(y)\|_+ \le c_1(1 + \|y\|_X^{p-1})\ \forall y \in X; \tag{7.113}$$

$$\exists C_2, C_3 > 0 : \quad [\hat{A}(y), y]_- \ge C_2 \|y\|_X^p - C_3\ \forall y \in X. \tag{7.114}$$

Previous results of this chapter and [49] provide the existence of a weak solution of Cauchy problem (7.110) with initial data

$$y(\tau) = y_\tau \tag{7.115}$$

on the interval $[\tau, T]$ for an arbitrary $y_\tau \in H$. Therefore, the next result takes place

Lemma 7.9. *$\forall \tau < T, y_\tau \in H$ Cauchy problem (7.110), (7.115) has a weak solution on the interval $[\tau, T]$. Moreover, each weak solution $u \in X_{\tau,T}$ of Cauchy problem (7.110), (7.115) on the interval $[\tau, T]$ belongs to $W_{\tau,T} \subset C([\tau, T]; H)$.*

Remark 7.2. Since $W_{\tau,T} \subset C([\tau, T]; H)$, for each weak solution of problem (7.110), in view of Lemma 7.9, initial data (7.115) has sense.

For fixed $\tau < T$, we denote

$$\mathscr{D}_{\tau,T}(u_\tau) = \{u(\cdot) \mid u \text{ is a weak solution of (7.110) on } [\tau, T],\ u(\tau) = u_\tau\}, \quad u_\tau \in H.$$

From Lemma 7.9, it follows that $\mathscr{D}_{\tau,T}(u_\tau) \neq \emptyset$ and $\mathscr{D}_{\tau,T}(u_\tau) \subset W_{\tau,T}\ \forall \tau < T,\ u_\tau \in H$.

From conditions for the parameters of problem (7.110) and Gronwall's lemma, it naturally follows the next result:

Lemma 7.10. *There exist $c_4, c_5, c_6, c_7, R_0 > 0$ such that for any finite interval of time $[\tau, T]$, every weak solution u of problem (7.110) on $[\tau, T]$ satisfies estimates: $\forall t \ge s,\ t, s \in [\tau, T]$*

$$\|u(t)\|_H^2 + c_4 \int_s^t \|u(\xi)\|_V^p d\xi \le \|u(s)\|_H^2 + c_5 \left(1 + \|f\|_{V^*}^2\right)(t-s), \tag{7.116}$$

$$\|u(t)\|_H^2 \le \|u(s)\|_H^2 e^{-c_6(t-s)} + c_7 \left(1 + \|f\|_{V^*}^2\right), \tag{7.117}$$

$$\|u(t)\|_H^2 - R_0^2 \le \left(\|u(s)\|_H^2 - R_0^2\right) e^{-c_6(t-s)}. \tag{7.118}$$

Analyzing the proof of Theorem 2.1 and Corollary 2.1, we get the next result. The similar results can be obtained for the second-order evolution inclusions.

Theorem 7.5. *Let $\tau < T$, $\{u_n\}_{n\ge 1}$ be an arbitrary sequence of weak solutions of (7.110) on $[\tau, T]$ such that $u_n(\tau) \to \eta$ weakly in H. Then, there exist $\{u_{n_k}\}_{k\ge 1} \subset \{u_n\}_{n\ge 1}$ and $u(\cdot) \in \mathscr{D}_{\tau,T}(\eta)$ such that*

$$\forall \varepsilon \in (0, T-\tau) \quad \max_{t\in[\tau+\varepsilon,T]} \|u_{n_k}(t) - u(t)\|_H \to 0, \quad k \to +\infty. \tag{7.119}$$

Moreover, if $u_n(\tau) \to \eta$ strongly in H, $n \to +\infty$, then up to a subsequence $u_{n_k} \to u$ in $C([\tau, T]; H)$, $k \to +\infty$.

For every $u_\tau \in H$, $\tau \in \mathbf{R}$, and $t \ge \tau$, we put

$$U(t,\tau,u_\tau) = \left\{ \tilde{u}(t) \;\middle|\; \begin{array}{l} \tilde{u}(\cdot) \text{ is a weak solution of the} \\ \text{problem (7.110) on } [\tau,T],\ \tilde{u}(\tau) = u_\tau \end{array} \right\}. \tag{7.120}$$

Now let us examine long-time behavior for the first-order evolution inclusions.

Theorem 7.6. *For the multivalued process U, given by (7.120), there exists a strictly invariant pullback attractor $\{\Theta(t)\}_{t\in\mathbf{R}}$ such that $\Theta(t) \subset B_{R_0}$, for all $t \in \mathbf{R}$.*

Proof. First of all, from (7.118), for every $R > R_0$, $u_\tau \in H$ such that $\|u_\tau\|_H \le R$, it holds

$$\|u(s+\tau)\|_H^2 - R_0^2 \le e^{-c_6(s+\tau)}\left(\|u_\tau\|_H^2 - R_0^2\right)e^{c_6\tau},$$
$$\|u(s+\tau)\|_H^2 \le e^{-c_6 s}(R^2 - R_0^2) + R_0^2.$$

So,

$$\sup_{\tau\in\mathbf{R}} \operatorname{dist}\bigl(U(s+\tau,\tau,B_R), B_{R_0}\bigr) \to 0, \text{ as } s \to +\infty. \tag{7.121}$$

In virtue of Theorem 6.1 and

$$U(t,s,B_R) \subset U(t,t-1,U(t-1,s,B_R)) \subset U(t,t-1,B_{R_0+1}),$$

where the last inclusion follows from (7.121) by taking a sufficiently small s, we only need to prove that the set $K(t) := \overline{U(t,t-1,B_{R_0+1})}$ is compact and that the map $x \mapsto U(t, \tau, x)$ has closed graph. These two properties are true, if the following statements holds for all $t \geq \tau$:

- $(U1)$ If $\eta_n \to \eta$ weakly in H and $\xi_n \in U(t, \tau, \eta_n)$, then the sequence $\{\xi_n\}$ is precompact in H.
- $(U2)$ If $\eta_n \to \eta$ strongly in H and $\xi_n \in U(t, \tau, \eta_n)$, then up to subsequence $\xi_n \to \xi \in U(t, \tau, \eta)$.

These properties are fulfilled due to Theorem 7.5.
The theorem is proved. □

7.5 Applications

As applications, we can consider all previous examples both in autonomous and nonautonomous cases for the first-order evolution inclusions. Obtained results allow us to study the dynamics of solutions of new classes of evolution equations of non-linear nonautonomous mathematical models of geophysical and socioeconomical processes and fields with interaction function of pseudomonotone type satisfying the condition of "no more than polynomial growth" and standard sign condition.

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Appendix A
Functional Spaces: The Embedding and Approximation Theorems

If it is necessary to describe a nonstationary process, which evolve in some domain Ω from some finite dimensional space $\mathbf{R}^n$ during the time interval S, we may deal with state and time functions, that is, with functions that put in correspondence for the each pair $\{t, x\} \in S \times \Omega$ the real number or vector $u(t, x)$. In virtue of this approach, the time and the space variables are equivalent. But there is one more convenient approach to the mathematical description for nonstationary processes [2, 3]: for each point in time t, it is mapped the state function $u(t, \cdot)$. (For example, for each point of time, we put the temperature distribution or velocity distribution in the domain Ω.)

geophysical process or field	$\Rightarrow$	mathematical model (partial differential equation)	$\Rightarrow$	differential-operator inclusion
		$y : \Omega \times [\tau, T] \to \mathbf{R}^m$		$y : [\tau, T] \to (\Omega \to \mathbf{R}^m)$ $y(\cdot, t) \in (\Omega \to \mathbf{R}^m)$, $t \in [\tau, T]$

Thus, we consider some functions, well-defined at the time interval S, with values from the state functions space (e.g., in the space $H_0^1(\Omega)$). Therefore, at studying some problems that depend on time, it is rather natural to consider some function spaces that acts from S into some infinite dimensional space X, in particular, it is natural to consider the spaces of integrable and differentiable functions. Further, we will consider only *real* linear spaces.

In this section, we introduce function spaces that will be used under investigation of differential-operator inclusions of such type in infinite-dimensional spaces:

$$Lu + A(u) + B(u) \ni f, \quad u \in D(L), \tag{A.1}$$

where $A : X_1 \to 2^{X_1^*}$, $B : X_2 \to 2^{X_2^*}$ are strict multivalued maps of $D(L)_{\lambda_0}$-pseudomonotone type with nonempty, convex, closed, bounded values, X_1, X_2 are

M.Z. Zgurovsky et al., *Evolution Inclusions and Variation Inequalities for Earth Data Processing III*, Advances in Mechanics and Mathematics 27,
DOI 10.1007/978-3-642-28512-7,

Banach spaces continuously embedded in some Hausdorff linear topological space, $X = X_1 \cap X_2$, $L : D(L) \subset X \to X^*$ is linear, monotone, closed, densely defined operator with a linear definitional domain $D(L)$. Moreover, we prove the important properties for this spaces. We consider constructions that guarantee the convergence of Faedo–Galerkin method for differential-operator inclusions with w_{λ_0}-pseudomonotone maps. The outcomes reduced with detailed proofs, because the given classes of spaces, in most cases, were not considered yet. Let us consider some results from [2, 3].

In the following referring to Banach spaces X, Y, when we write

$$X \subset Y$$

we mean the embedding in the set-theory sense and in the topological sense.

For the very beginning, let us consider ideas of the sum and intersection of Banach spaces required for studying of anisotropic problems.

For $n \geq 2$, let $\{X_i\}_{i=1}^n$ be some family of Banach spaces.

Definition A.1. *The interpolation family* refers a family of Banach spaces $\{X_i\}_{i=1}^n$ such that for some linear topological space (LTS) Y, we have

$$X_i \subset Y \quad \text{for all} \quad i = \overline{1,n}.$$

As $n = 2$, the interpolation family is called *the interpolation pair*.

Further, let $\{X_i\}_{i=1}^n$ be some interpolation family. On the analogy of ([1, p.23]), in the linear variety $X = \cap_{i=1}^n X_i$, we consider the norm

$$\|x\|_X := \sum_{i=1}^n \|x\|_{X_i} \qquad \forall x \in X, \tag{A.2}$$

where $\|\cdot\|_{X_i}$ is the norm in X_i.

Proposition A.1. *Let $\{X, Y, Z\}$ be an interpolation family. Then,*

$$X \cap (Y \cap Z) = (X \cap Y) \cap Z = X \cap Y \cap Z, \quad X \cap Y = Y \cap X$$

both in the sense of equality of sets and in the sense of equality of norms.

We also consider the linear space

$$Z := \sum_{i=1}^n X_i = \left\{ \sum_{i=1}^n x_i \;\middle|\; x_i \in X_i,\ i = \overline{1,n} \right\}$$

with the norm

$$\|z\|_Z := \inf \left\{ \max_{i=\overline{1,n}} \|x_i\|_{X_i} \;\middle|\; x_i \in X_i,\ \sum_{i=1}^{n} x_i = z \right\} \qquad \forall z \in Z. \tag{A.3}$$

Proposition A.2. *Let $\{X_i\}_{i=1}^n$ be an interpolation family. Then $X = \cap_{i=1}^n X_i$ and $Z = \sum_{i=1}^n X_i$ are Banach spaces and it results in*

$$X \subset X_i \subset Z \quad \textit{for all} \quad i = \overline{1,n}. \tag{A.4}$$

Remark A.1. Let Banach spaces X and Y satisfy the following conditions:

$$X \subset Y, \qquad X \quad \text{is dense in} \quad Y,$$
$$\|x\|_Y \le \gamma \|x\|_X \quad \forall x \in X, \qquad \gamma = \text{const}.$$

Then,

$$Y^* \subset X^*, \qquad \|f\|_{X^*} \le \gamma \|f\|_{Y^*} \quad \forall f \in Y^*.$$

Moreover, if X is reflexive, then Y^* is dense in X^*.

Let $\{X_i\}_{i=1}^n$ be an interpolation family such that the space $X := \cap_{i=1}^n X_i$ with the norm (A.2) is dense in X_i for all $i = \overline{1,n}$. Due to Remark A.1, the space X_i^* may be considered as subspace of X^*. Thus, we can construct $\sum_{i=1}^n X_i^*$ and

$$\sum_{i=1}^{n} X_i^* \subset \left(\bigcap_{i=1}^{n} X_i \right)^* . \tag{A.5}$$

Under the given assumptions X is dense in $Z := \sum\limits_{i=1}^{n} X_i$ for every $i = \overline{1,n}$. So X_i is dense in Z too. Thanks to Remark A.1, we can consider space Z^* as a subspace of X_i^* for all $i = \overline{1,n}$, and also as a subspace of $\cap_{i=1}^n X_i^*$, that is,

$$\left(\sum_{i=1}^{n} X_i \right)^* \subset \bigcap_{i=1}^{n} X_i^* . \tag{A.6}$$

Proposition A.3. *Let $\{X_i\}_{i=1}^n$ be an interpolation family such that the space $X := \cap_{i=1}^n X_i$ with the norm (A.2) is dense in X_i for all $i = \overline{1,n}$. Then*

$$\sum_{i=1}^{n} X_i^* = \left(\bigcap_{i=1}^{n} X_i \right)^*$$

and

$$\left(\sum_{i=1}^{n} X_i \right)^* = \bigcap_{i=1}^{n} X_i^*$$

both in the sense of sets equality and in the sense of the equality of norms.

Theorem A.1. *(The reflexivity criterium) Banach space E is reflexive if and only if each bounded in E sequence contains the weakly convergent in E subsequence.*

Theorem A.2. *(Banach–Alaoglu) In reflexive Banach space, each bounded sequence contains the weakly convergent subsequence.*

Now, let us consider classes of integrable by Bochner distributions with values in Banach space. Such extended phase spaces generally appear when studying evolutional geophysical processes and fields.

Now, let Y be some Banach space, Y^* its topological adjoint space, S be some compact time interval. We consider the classes of functions defined on S and imagines in Y (or in Y^*).

The set $L_p(S;Y)$ of all measured by Bochner functions (see [1]) as $1 \le p \le +\infty$ with the natural linear operations with the norm

$$\|y\|_{L_p(S;Y)} = \left(\int_S \|y(t)\|_Y^p \, dt \right)^{1/p}$$

is a Banach space. As $p = +\infty$ $L_\infty(S;Y)$ with the norm

$$\|y\|_{L_\infty(S;Y)} = \operatorname*{vrai\,max}_{t \in S} \|y(t)\|_Y$$

is a Banach space.

The next theorem shows that under the assumption of reflexivity and separability of Y the adjoint with $L_p(S;Y)$, $1 \le p < +\infty$, space $(L_p(S;Y))^*$ may be identified with $L_q(S;Y^*)$, where q is such that $p^{-1} + q^{-1} = 1$.

Theorem A.3. *If the space Y is reflexive and separable and $1 \le p < +\infty$, then each element $f \in (L_p(S;Y))^*$ has the unique representation*

$$f(y) = \int_S \langle \xi(t), y(t) \rangle_Y \, dt \quad \textit{for every } y \in L_p(S;Y)$$

with the function $\xi \in L_q(S;Y^*)$, $p^{-1} + q^{-1} = 1$. The correspondence $f \to \xi$, with $f \in (L_p(S;Y))^*$ is linear and

$$\|f\|_{(L_p(S;Y))^*} = \|\xi\|_{L_q(S;Y^*)}.$$

Now, let us consider the reflexive separable Banach space V with the norm $\|\cdot\|_V$ and the Hilbert space $(H, (\cdot,\cdot)_H)$ with the norm $\|\cdot\|_H$, and for the given spaces, let the next conditions be satisfied

$$\begin{gathered} V \subset H, \quad V \text{ is dense in } H, \\ \exists \gamma > 0 \; : \; \|v\|_H \le \gamma \|v\|_V \quad \forall v \in V. \end{gathered} \tag{A.7}$$

Due to Remark A.1 under the given assumptions, we may consider the adjoint with H space H^* as a subspace of V^* that is adjoint with V. As V is reflexive, then H^* is dense in V^* and

$$\|f\|_{V^*} \le \gamma \|f\|_{H^*} \quad \forall f \in H^*,$$

where $\|\cdot\|_{V^*}$ and $\|\cdot\|_{H^*}$ are the norm in V^* and in H^*, respectively.

Further, we identify the spaces H and H^*. Then we obtain

$$V \subset H \subset V^*$$

with continuous and dense embedding.

Definition A.2. The triple of spaces $(V; H; V^*)$ that satisfy the latter conditions will be called *the evolution triple*.

Let us point out that under identification H with H^* and H^* with some subspace of V^*, an element $y \in H$ is identified with some $f_y \in V^*$ and

$$(y, x) = \langle f_y, x \rangle_V \quad \forall x \in V,$$

where $\langle \cdot, \cdot \rangle_V$ is the canonical pairing between V^* and V. Since the element y and f_y are identified, then under condition (A.7), the pairing $\langle \cdot, \cdot \rangle_V$ will denote the inner product on H $(\cdot, \cdot)$.

We consider $p_i,\ r_i,\ i = 1, 2$ such that $1 < p_i \le r_i \le +\infty,\ p_i < +\infty$. Let $q_i \ge r_i' \ge 1$ well defined by

$$p_i^{-1} + q_i^{-1} = r_i^{-1} + r_i'^{-1} = 1 \qquad \forall i = 1, 2.$$

Remark that $1/\infty = 0$.

Now, we consider some Banach spaces that play an important role in the investigation of the differential-operator equations and evolution variation inequalities in nonreflexive Banach spaces.

Referring to evolution triples $(V_i; H; V_i^*)$ $(i = 1, 2)$ such that

$$\text{the set } V_1 \cap V_2 \text{ is dense in the spaces } V_1,\ V_2, \text{ and } H, \tag{A.8}$$

we consider the functional Banach spaces (Proposition A.2)

$$X_i = X_i(S) = L_{q_i}(S; V_i^*) + L_{r_i'}(S; H), \quad i = 1, 2$$

with norms

$$\|y\|_{X_i} = \inf\Big\{ \max\big\{ \|y_1\|_{L_{q_i}(S;V_i^*)};\ \|y_2\|_{L_{r_i'}(S;H)} \big\} \ \Big|\ y_1 \in L_{q_i}(S; V_i^*),\, y_2 \in L_{r_i'}(S; H),\ y = y_1 + y_2 \Big\},$$

for all $y \in X_i$, and

$$X = X(S) = L_{q_1}(S; V_1^*) + L_{q_2}(S; V_2^*) + L_{r_2'}(S; H) + L_{r_1'}(S; H)$$

with

$$\|y\|_X = \inf\Big\{ \max_{i=1,2} \big\{ \|y_{1i}\|_{L_{q_i}(S;V_i^*)};\ \|y_{2i}\|_{L_{r_i'}(S;H)} \big\} \,\Big|\, y_{1i} \in L_{q_i}(S; V_i^*),$$
$$y_{2i} \in L_{r_i'}(S; H),\ i = 1, 2;\ y = y_{11} + y_{12} + y_{21} + y_{22}\Big\},$$

for each $y \in X$. As $r_i < +\infty$, due to Theorem A.3 and to Theorem A.3, the space X_i is reflexive. Analogously, if $\max\{r_1, r_2\} < +\infty$, the space X is reflexive.

Under the latter theorems, we identify the adjoint with $X_i(S)$, $X_i^* = X_i^*(S)$, with $L_{r_i}(S; H) \cap L_{p_i}(S; V_i)$, where

$$\|y\|_{X_i^*} = \|y\|_{L_{r_i}(S;H)} + \|y\|_{L_{p_i}(S;V_i)} \quad \forall y \in X_i^*,$$

and, respectively, the adjoint with $X(S)$ space $X^* = X^*(S)$ we identify with

$$L_{r_1}(S; H) \cap L_{r_2}(S; H) \cap L_{p_1}(S; V_1) \cap L_{p_2}(S; V_2),$$

where

$$\|y\|_{X^*(S)} = \|y\|_{L_{r_1}(S;H)} + \|y\|_{L_{r_2}(S;H)} + \|y\|_{L_{p_1}(S;V_1)} + \|y\|_{L_{p_2}(S;V_2)} \quad \forall y \in X^*.$$

On $X(S) \times X^*(S)$, we denote by

$$\langle f, y\rangle = \langle f, y\rangle_S = \int_S (f_{11}(\tau), y(\tau))_H\, d\tau + \int_S (f_{12}(\tau), y(\tau))_H\, d\tau$$
$$+ \int_S \langle f_{21}(\tau), y(\tau)\rangle_{V_1}\, d\tau + \int_S \langle f_{22}(\tau), y(\tau)\rangle_{V_2}\, d\tau$$
$$= \int_S (f(\tau), y(\tau))\, d\tau \quad \forall f \in X,\ y \in X^*,$$

where $f = f_{11} + f_{12} + f_{21} + f_{22}$, $f_{1i} \in L_{r_i'}(S; H)$, $f_{2i} \in L_{q_i}(S; V_i^*)$, $i = 1, 2$.

In that case when $\max\{r_1, r_2\} < +\infty$, with corresponding to the "standard" denotations [1, p. 171], the spaces X^*, X_1^* and X_2^* further will denote as X, X_1, and X_2 respectively, and vice verse, X, X_1, and X_2 as X^*, X_1^*, and X_2^*, respectively. The given renotation is correct in virtue of the next proposition, that is the direct corollary of Proposition A.3 and of Theorem A.3.

Proposition A.4. *As* $\max\{r_1, r_2\} < +\infty$*, the spaces* X*,* X_1 *and* X_2 *are reflexive.*

Let $\mathscr{D}(S)$ be the space of the principal functions on S. For a Banach space X as $\mathscr{D}^*(S; X)$, we will denote the family of all linear continuous maps from $\mathscr{D}(S)$ into X, with the weak topology. The elements of the given space are called the distributions on S with values in X. For each $f \in \mathscr{D}^*(S; X)$, the generalized derivative is well defined by the rule

$$f'(\varphi) = -f(\varphi') \quad \forall \varphi \in \mathscr{D}(S).$$

We remark that each locally integrable in Bochner sense function u, we can identify with corresponding distribution $f_u \in \mathscr{D}^*(S; X)$ in such way:

$$f_u(\varphi) = u(\varphi) = \int_S u(t)\varphi(t)dt \quad \forall \varphi \in \mathscr{D}(S), \tag{A.9}$$

where the integral is regard in the Bochner sense. We will interpret the family of all locally Bochner integrable functions from $(S \to X)$ as subspace in $\mathscr{D}^*(S; X)$. Thus, the distributions, that allow the representation (A.9), we will consider as functions from $(S \to X)$. We are also remark that the correspondence $\mathscr{D}^*(S; X) \ni f \to f' \in \mathscr{D}^*(S; X)$ is continuous [1, p.169].

Definition A.3. As $C^m(S; X)$, $m \geq 0$, we refer the family of all functions from $(S \to X)$ that have the continuous derivatives by the order m inclusively. In that case, when S is a compact interval, $C^m(S; X)$ is a Banach space with the norm

$$\|y\|_{C^m(S;X)} = \sum_{i=0}^{m} \sup_{t \in S} \left\| y^{(i)}(t) \right\|_X ,$$

where $y^{(i)}(t)$ is the strong derivative from y at the point $t \in S$ by the order $i \geq 1$; $y^{(0)} \equiv y$.

Let $V = V_1 \cap V_2$. Then $V^* = V_1^* + V_2^*$. At studying the differential-operator inclusions and evolution variation inequalities together with the spaces X and X^*, one more space, which we will denote as $W^* = W^*(S)$, plays the important role. Let us set

$$W^*(S) = \{y \in X^*(S) |\ y' \in X(S)\},$$

where the derivative y' from $y \in X^*$ is considered in the sense of scalar distributions space $\mathscr{D}^*(S; V^*)$.

The class W^* generally contains the generalized solutions of the first-order differential-operator equations and inclusions with maps of pseudomonotone type. By analogy with Sobolev spaces, it is required to study some structured properties, embedding and approximations theorems, as well as some "rules of work" with elements of such spaces.

Theorem A.4. *The set W^* with the natural operations and graph norm for y':*

$$\|y\|_{W^*} = \|y\|_{X^*} + \|y'\|_X \quad \forall y \in W^*$$

is Banach space.

Theorem A.5. *The set $C^1(S;V) \cap W_0^*$ is dense in W_0^*.*

Theorem A.6. *$W_0^* \subset C(S;H)$ with continuous embedding. Moreover, for every $y, \xi \in W_0^*$ and $s, t \in S$ the next formula of integration by parts takes place*

$$\big(y(t), \xi(t)\big) - \big(y(s), \xi(s)\big) = \int_s^t \Big\{(y'(\tau), \xi(\tau)) + (y(\tau), \xi'(\tau))\Big\} d\tau. \tag{A.10}$$

In particular, when $y = \xi$ we have

$$\frac{1}{2}\Big(\|y(t)\|_H^2 - \|y(s)\|_H^2\Big) = \int_s^t (y'(\tau), y(\tau)) d\tau. \tag{A.11}$$

Corollary A.1. *$W^* \subset C(S;H)$ with continuous embedding. Moreover, for every $y, \xi \in W^*$ and $s, t \in S$ formula (A.10) takes place.*

Remark A.2. When $\max\{r_1, r_2\} < +\infty$, due to the standard denotations [1, p. 173], we will denote the space W^* as W; "*" will direct on nonreflexivity of the spaces X and W.

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M.Z. Zgurovsky et al., *Evolution Inclusions and Variation Inequalities for Earth Data Processing III*, Advances in Mechanics and Mathematics 27,
DOI 10.1007/978-3-642-28512-7, © Springer-Verlag Berlin Heidelberg 2012